MIDTERM SOLUTIONS - MATH 2121, FALL 2019.

Name:

Student ID:

Email:

Lecture: L1 (Wed/Fri) L2 (Mon/Wed)

Tutorial: T1A T1B T1C T2A T2B T2C

Problem #	Max points possible	Actual score
1	10	
2	10	
3	20	
4	10	
5	10	
6	10	
7	10	
Total	80	

You have **120 minutes** to complete this exam.

No books, notes, or electronic devices can be used on the test.

Draw a $\boxed{\text{box}}$ around your answers or write your answers in the $\boxed{\text{boxes}}$ provided.

Partial credit can be given on some problems if you show your work. Good luck!

Problem 1. (10 points) Find all solutions to the linear system

$$\begin{aligned} x_1 - x_2 - 6x_3 &= 10 \\ 2x_2 + 7x_3 &= -10 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

Solution:

The linear system has augmented matrix $A = \begin{bmatrix} 1 & -1 & -6 & 10 \\ 0 & 2 & 7 & -10 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

We can convert this to reduced echelon form by the row operations

$$\begin{aligned} A = \begin{bmatrix} 1 & -1 & -6 & 10 \\ 0 & 2 & 7 & -10 \\ 1 & 1 & 1 & 0 \end{bmatrix} &\xrightarrow{\text{subtract row 1 from row 3}} \begin{bmatrix} 1 & -1 & -6 & 10 \\ 0 & 2 & 7 & -10 \\ 0 & 2 & 7 & -10 \end{bmatrix} \\ &\xrightarrow{\text{subtract row 2 from row 3}} \begin{bmatrix} 1 & -1 & -6 & 10 \\ 0 & 2 & 7 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{\text{multiple row 2 by 1/2}} \begin{bmatrix} 1 & -1 & -6 & 10 \\ 0 & 1 & 7/2 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{\text{add row 2 to row 1}} \begin{bmatrix} 1 & 0 & -5/2 & 5 \\ 0 & 1 & 7/2 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A). \end{aligned}$$

The last matrix does not have a pivot in the last column, so our original system has at least one solution. Since only columns 1 and 2 contain pivots, we conclude that x_1 and x_2 are basic variables while x_3 is a free variable. The two nontrivial equations in the linear system whose augmented matrix in $\mathsf{RREF}(A)$ expresses the basic variables in terms of the free variables:

$$x_1 - \tfrac{5}{2}x_3 = 5 \quad \text{and} \quad x_2 + \tfrac{7}{2}x_3 = -5.$$

We can choose any value a for x_3 and this determines x_1 and x_2 via these equations. Thus the solutions to the original system are given by all triples

$$\boxed{(x_1, x_2, x_3) = (5 + \tfrac{5}{2}a, -5 - \tfrac{7}{2}a, a)}$$

where $a \in \mathbb{R}$ ranges over all real numbers.

Problem 2. (10 points)

Find all values of a such that $\left\{ \begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} a+2 \\ a+6 \end{bmatrix} \right\}$ is linearly independent in $\mathbb{R}^2$.

Solution:

The vectors are linearly independent if the matrix $A = \begin{bmatrix} 1 & a+2 \\ a & a+6 \end{bmatrix}$ has a pivot in every column, which occurs only if $\mathsf{RREF}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ since A is square.

But if try to row reduce A we get

$$A = \begin{bmatrix} 1 & a+2 \\ a & a+6 \end{bmatrix} \xrightarrow{\text{add } -a \text{ times row 1 to row 2}} \begin{bmatrix} 1 & a+2 \\ 0 & a+6-a(a+2) \end{bmatrix} = \begin{bmatrix} 1 & a+2 \\ 0 & 6-a-a^2 \end{bmatrix}.$$

If $6-a-a^2 \neq 0$ then two further row operations (first rescale row 2 and then subtract a multiple of row 2 from row 1) will transform the last matrix to the identity matrix. Therefore if $6-a-a^2 \neq 0$ then the vectors are linearly independent.

On the other hand, if $6-a-a^2 = 0$ then A evidently has only one pivot column so the vectors must be linearly dependent.

Since $6-a-a^2 = (3+a)(2-a)$, we have $6-a-a^2 = 0$ if and only if $a = -3$ or $a = 2$. We conclude that the vectors are linearly independent for

$$\boxed{\text{all values of } a \text{ with } a \neq -3 \text{ and } a \neq 2.}$$

Another way to solve the problem: the vectors are linearly independent if and only if $\det A \neq 0$. Since $\det A = (a+6) - a(a+2) = 6 - a - a^2 = (3+a)(2-a)$, we reach the same answer as before.

Problem 3. (20 points) Indicate which of the following is TRUE or FALSE.

(1) An invertible matrix can have more than one echelon form.
(2) Suppose U and V are subspaces of $\mathbb{R}^2$. If $\dim U < \dim V$ then $U \subset V$.
(3) Suppose U and V are subspaces of $\mathbb{R}^3$. If $\dim U < \dim V$ then $U \subset V$.
(4) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and onto then $n \geq m$.
(5) Four vectors in $\mathbb{R}^3$ can be linearly independent if they are all nonzero.
(6) If $\det A = \pm 1$, then A must be a permutation matrix.
(7) If $a + d + g = b + e + h = c + f + i = 0$ then $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is not invertible.
(8) Suppose A and B are matrices such that AB is defined. If AB is invertible then A and B are either both invertible or both not invertible.
(9) The inverse of a permutation matrix is the same as its transpose.
(10) If two rows of a square matrix A are the same then $\det A = 0$.

Each part will be graded as follows: 0 points for a wrong or missing answer, 2 points for the correct answer. Explanations are not required for answers.

Solution:

(1)	$\boxed{\text{TRUE}}$	FALSE
(2)	$\boxed{\text{TRUE}}$	FALSE
(3)	TRUE	$\boxed{\text{FALSE}}$
(4)	$\boxed{\text{TRUE}}$	FALSE
(5)	TRUE	$\boxed{\text{FALSE}}$
(6)	TRUE	$\boxed{\text{FALSE}}$
(7)	$\boxed{\text{TRUE}}$	FALSE
(8)	$\boxed{\text{TRUE}}$	FALSE
(9)	$\boxed{\text{TRUE}}$	FALSE
(10)	$\boxed{\text{TRUE}}$	FALSE

Solution:

(1) $\boxed{\text{TRUE}}$ FALSE

$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is an echelon form of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for all $a, c \neq 0$.

(2) $\boxed{\text{TRUE}}$ FALSE

All subspaces of $\mathbb{R}^2$ have dimensions 0, 1, or 2. Only $\{0\}$ has dimension 0 and only $\mathbb{R}^2$ has dimension 2. Any subspace of dimension 1 contains $\{0\}$ and is contained in $\mathbb{R}^2$.

(3) TRUE $\boxed{\text{FALSE}}$

Then line $U = \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$ is not contained in the plane $V \left\{ \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\}$. But U, V are subspaces of $\mathbb{R}^2$ with $\dim U = 1 < 2 = \dim V$.

(4) $\boxed{\text{TRUE}}$ FALSE

We proved in class that if $n \leq m$ then T cannot be onto and linear.

(5) TRUE $\boxed{\text{FALSE}}$

If $p > n$ then any p vectors in $\mathbb{R}^n$ act linearly dependent.

(6) TRUE $\boxed{\text{FALSE}}$

The triangular matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ also determinant ± 1.

(7) $\boxed{\text{TRUE}}$ FALSE

The matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is invertible if and only if A^T is invertible.

But if $a + d + g = b + e + h = c + f + i = 0$ then $A^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$, so the columns of A^T are not linearly independent so A^T is not invertible.

(8) $\boxed{\text{TRUE}}$ FALSE

If X and Y are $n \times n$ matrices then $XY = I_n$ implies $YX = I_n$. This does not hold if the matrices are not square. Since AB is defined, we know that A is $m \times n$ and B and $n \times p$ and AB is $m \times p$ for some numbers m, n, p.

If AB and A are invertible then AB and A are both square, so $m = n = p$ and B is also square, so B is invertible with inverse $B^{-1} = (AB)^{-1}A$, as

$$(AB)^{-1}A \cdot B = (AB)^{-1}(AB) = I_n.$$

If AB and B are invertible then AB and B are both square, so $m = n = p$ and A is also square, so A is invertible with inverse $A^{-1} = B(AB)^{-1}$, as

$$A \cdot B(AB)^{-1} = (AB)(AB)^{-1} = I_n.$$

Therefore, if AB is invertible, then it is not possible for A but not B to be invertible, or for B but not A to be invertible.

(9) $\boxed{\text{TRUE}}$ FALSE

If $e_1, e_2, \dots, e_n$ is the standard basis of $\mathbb{R}^n$ then $e_i^T e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

Suppose X is an $n \times n$ permutation matrix. Then $X = \begin{bmatrix} e_{i_1} & e_{i_2} & \dots & e_{i_n} \end{bmatrix}$ where $i_1, i_2, \dots, i_n$ are the numbers $1, 2, \dots, n$ arranged in some order.

The entry in position (j, k) of

$$X^T X = \begin{bmatrix} e_{i_1}^T \\ e_{i_2}^T \\ \vdots \\ e_{i_n}^T \end{bmatrix} \begin{bmatrix} e_{i_1} & e_{i_2} & \dots & e_{i_n} \end{bmatrix}$$

is therefore $e_{i_j}^T e_{i_k}$ which is 1 if $j = k$ and 0 if $j \neq k$. Thus $X^T X = I_n$.

(10) $\boxed{\text{TRUE}}$ FALSE

If two rows of a square matrix A are the same, then two columns of A^T are the same, so $0 = \det A^T = \det A$.

Problem 4. (10 points)

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 5 \\ 8 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 9 \end{bmatrix}.$$

Find the standard matrix of T.
In other words, find a 2×2 matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^2$.

Solution:

If a, b, c, d are any numbers with $ad - bc \neq 0$ then

$$\frac{1}{ad-bc}\left(d\begin{bmatrix} a \\ b \end{bmatrix} - b\begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\frac{1}{ad-bc}\left(-c\begin{bmatrix} a \\ b \end{bmatrix} + a\begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Things are a little simpler in the problem at hand, since can just write

$$8\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad 2\begin{bmatrix} 5 \\ 8 \end{bmatrix} - 5\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

By linearity, we have

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 8T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) - 3T\left(\begin{bmatrix} 5 \\ 8 \end{bmatrix}\right) = 8\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} -4 \\ -11 \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -5T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 5 \\ 8 \end{bmatrix}\right) = -5\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

The standard matrix of T is therefore

$$\boxed{\begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -11 & 8 \end{bmatrix}.}$$

Problem 5. (10 points) Consider the matrix

$$A = \begin{bmatrix} 5 & 5 & 6 & 6 & 7 \\ 4 & 0 & 4 & 4 & 0 \\ 3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 5 & 6 & 7 & 1 & 8 \end{bmatrix}.$$

(a) Compute A^{-1} or explain why A is not invertible.

(b) Compute $\det A$.

Solution:

This was a difficult problem requiring many careful computations. To determine if A is invertible and at the same time compute A^{-1}, we can try to row reduce

$$B = \left[\begin{array}{ccccc|ccccc} 5 & 5 & 6 & 6 & 7 & 1 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & 4 & 4 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 5 & 6 & 7 & 1 & 8 & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right].$$

I have drawn $\cdot$ instead of 0 to reduce the amount of writing necessary. Here is one possible sequence of matrices that are row equivalent to B:

$$(1) \quad \left[\begin{array}{ccccc|ccccc} 3 & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 5 & 6 & 7 & 1 & 8 & \cdot & \cdot & \cdot & \cdot & 1 \\ 4 & \cdot & 4 & 4 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 5 & 5 & 6 & 6 & 7 & 1 & \cdot & \cdot & \cdot & \cdot \end{array}\right].$$

$$(2) \quad \left[\begin{array}{ccccc|ccccc} 3 & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & -5 & 1 & -1 & \cdot & \cdot & \cdot & 1 \\ 4 & \cdot & 4 & 4 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 5 & 5 & 6 & 6 & 7 & 1 & \cdot & \cdot & \cdot & \cdot \end{array}\right].$$

$$(3) \quad \left[\begin{array}{ccccc|ccccc} 3 & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & 1 & 1 & -1 & \cdot & \cdot & 3 & 1 \\ 4 & \cdot & 4 & 4 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 5 & 5 & 6 & 6 & 7 & 1 & \cdot & \cdot & \cdot & \cdot \end{array}\right].$$

$$(4) \quad \left[\begin{array}{ccccc|ccccc} 3 & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & 1 & 1 & -1 & \cdot & \cdot & 3 & 1 \\ 4 & \cdot & 4 & 4 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 5 & \cdot & 1 & 1 & 2 & 6 & \cdot & \cdot & -15 & -5 \end{array}\right].$$

$$(5)\quad \left[\begin{array}{ccccc|ccccc} 3 & . & . & . & . & . & . & 1 & -3/2 & . \\ . & 1 & 1 & . & 1 & -1 & . & . & 5/2 & 1 \\ 4 & . & 4 & . & . & . & 1 & . & -2 & . \\ . & . & . & 1 & . & . & . & . & 1/2 & . \\ 5 & . & 1 & . & 2 & 6 & . & . & -31/2 & -5 \end{array}\right].$$

$$(6)\quad \left[\begin{array}{ccccc|ccccc} 1 & . & . & . & . & . & . & 1/3 & -1/2 & . \\ . & 1 & 1 & . & 1 & -1 & . & . & 5/2 & 1 \\ . & . & 4 & . & . & . & 1 & -4/3 & . & . \\ . & . & . & 1 & . & . & . & . & 1/2 & . \\ . & . & 1 & . & 2 & 6 & . & -5/3 & -13 & -5 \end{array}\right].$$

$$(7)\quad \left[\begin{array}{ccccc|ccccc} 1 & . & . & . & . & . & . & 1/3 & -1/2 & . \\ . & 1 & . & . & 1 & -1 & -1/4 & 1/3 & 5/2 & 1 \\ . & . & 1 & . & . & . & 1/4 & -1/3 & . & . \\ . & . & . & 1 & . & . & . & . & 1/2 & . \\ . & . & . & . & 2 & 6 & -1/4 & -4/3 & -13 & -5 \end{array}\right].$$

$$(8)\quad \left[\begin{array}{ccccc|ccccc} 1 & . & . & . & . & . & . & 1/3 & -1/2 & . \\ . & 1 & . & . & . & -4 & -1/8 & 1 & 9 & 7/2 \\ . & . & 1 & . & . & . & 1/4 & -1/3 & . & . \\ . & . & . & 1 & . & . & . & . & 1/2 & . \\ . & . & . & . & 1 & 3 & -1/8 & -2/3 & -13/2 & -5/2 \end{array}\right].$$

Since the first five columns are I_5, we conclude that A is invertible with

$$\boxed{A^{-1} = \left[\begin{array}{ccccc} 0 & 0 & 1/3 & -1/2 & 0 \\ -4 & -1/8 & 1 & 9 & 7/2 \\ 0 & 1/4 & -1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 3 & -1/8 & -2/3 & -13/2 & -5/2 \end{array}\right]}.$$

The determinant of A is easier to compute.

There are only two permutation matrices X with $\mathsf{prod}(X, A) \neq 0$:

$$\left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

The first permutation matrix X has $\mathsf{inv}(X) = 2$ while the second has $\mathsf{inv}(X) = 7$. The values of $\mathsf{prod}(X, A)$ are respectively $2 \cdot 3 \cdot 4 \cdot 5 \cdot 8$ and $2 \cdot 3 \cdot 4 \cdot 6 \cdot 7$, so

$$\det A = \sum_{X \in S_5} \mathsf{prod}(X, A)(-1)^{\mathsf{inv}(X)} = 2\cdot3\cdot4\cdot5\cdot8 - 2\cdot3\cdot4\cdot6\cdot7 = 24(40-42) = \boxed{-48}.$$

Problem 6. (10 points) Let m and n be positive integers.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard matrix A.
Recall that this means that A is a matrix such that $T(v) = Av$ for all $v \in \mathbb{R}^n$.

(a) How many rows does A have? How many columns does A have?

$\boxed{A \text{ has } m \text{ rows and } n \text{ columns.}}$

(b) If T is one-to-one, then what is the dimension of the column space of A? Explain your answer to receive full credit.

If T is one-to-one then the columns of A are linearly independent, so these n columns are a basis for the column space of A which has dimension n:

$$\boxed{\dim \mathrm{Col} A = n}.$$

(c) If T is onto, then what is the dimension of the null space of A? Explain your answer to receive full credit.

If T is onto then the column space of A is equal to $\mathbb{R}^m$ so has dimension m. By the Rank-Nullity theorem, we know that $n = \dim \mathrm{Col} A + \dim \mathrm{Nul} A$ so it follows that the null space of A has dimension $n - m$:

$$\boxed{\dim \mathrm{Nul} A = n - m}.$$

Problem 7. (10 points)

Find a basis for $\mathbb{R}^4$ that includes the vectors $u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

In other words, find vectors $w, x \in \mathbb{R}^4$ such that u, v, w, x is a basis for $\mathbb{R}^4$.
Justify your answer to receive full credit.

Solution:

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since the last four columns are the standard basis of $\mathbb{R}^4$, we have Col $= \mathbb{R}^4$. The pivot columns of A therefore be a basis for $\mathbb{R}^4$. Moreover, if the vectors u and v are linearly indepenedent they will be among the pivot columns of A, so this basis will include u and v.

We row reduce A to find its pivot columns as follows:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{bmatrix}.$$

The last matrix is in echelon form (though not reduced); its pivot positions are in columns 1, 2, 3, and 5. Therefore one basis for $\mathbb{R}^4$ containing u and v is

$$\boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.}$$