MIDTERM - MATH 2121, FALL 2017.

Name:

Email:

Problem #	Max points possible	Actual score
1	15	
2	20	
3	15	
4	20	
5	20	
6	10	
Total	100	

You have **120 minutes** to complete this exam.

No books, notes, or electronic devices can be used on the test.

Clearly label your answers by putting them in a box.

Partial credit can be given on some problems if you show your work. Good luck!

Problem 1. (2 + 9 + 4 = 15 points)

(a) Write down the augmented matrix of the linear system

$$\begin{aligned} 2x_1 + x_3 + 21x_4 &= 5 \\ x_1 + 9x_2 + 8x_3 + 3x_4 &= 6 \\ 3x_1 - 12x_4 &= 0. \end{aligned}$$

(b) Compute the reduced echelon form of the matrix in (a).

(c) How many solutions does our linear system

$$\begin{aligned} 2x_1 + x_3 + 21x_4 &= 5 \\ x_1 + 9x_2 + 8x_3 + 3x_4 &= 6 \\ 3x_1 - 12x_4 &= 0 \end{aligned}$$

have? To receive full credit, explain how you derive your answer.

Problem 2. (2 + 2 + . . . + 2 = 20 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Let A be a matrix. Indicate which of the following is TRUE or FALSE.

(1) If T is linear then its range is equal to its codomain.
(2) If T is linear and one-to-one then $n \geq m$.
(3) If T is linear and onto then $n \geq m$.
(4) If T is linear and invertible then $n \geq m$.
(5) If T is linear and invertible, then its inverse is linear and invertible.
(6) If T is linear, and A is its standard matrix, then A has size $n \times m$.
(7) If T is linear, and A is its standard matrix, and $T(v) = 0$ for a nonzero vector $v \in \mathbb{R}^n$, then the columns of A are not linearly independent.
(8) If B is a matrix with the same size as A and $Av = Bv$ whenever v is a vector such that the products Av and Bv are both defined, then $A = B$.
(9) If A is not invertible, then AB is not the identity matrix for any matrix B.
(10) If T is linear, and A is its standard matrix, and the range of T is not all of $\mathbb{R}^m$, then not every column of A is a pivot column.

Each part will be graded as follows: 0 points for a wrong answer, 1 point for no answer, 2 points for the correct answer. Explanations are not required for answers.

Solution:

(1)	TRUE	FALSE
(2)	TRUE	FALSE
(3)	TRUE	FALSE
(4)	TRUE	FALSE
(5)	TRUE	FALSE
(6)	TRUE	FALSE
(7)	TRUE	FALSE
(8)	TRUE	FALSE
(9)	TRUE	FALSE
(10)	TRUE	FALSE

Problem 3. (5 + 10 = 15 points)

(a) For what values of x is the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & x & 0 \\ -1 & 5 & 0 & 8 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

invertible?

(b) Assuming x is such that

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & x & 0 \\ -1 & 5 & 0 & 8 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

is invertible, derive a formula for A^{-1}.

Problem 4. (1 + 9 + 4 + 4 + 2 = 20 points) Consider the matrix

$$A = \begin{bmatrix} 5 & -1 & 6 & -4 \\ 2 & 3 & 16 & -5 \\ 0 & 2 & 8 & -2 \end{bmatrix}.$$

Remember that

- The *column space* of A is the span of its columns.
- The *null space* of A is the set of vectors v with $Av = 0$.

(a) What are the values of m and n such that $\mathrm{Col}A \subset \mathbb{R}^m$ and $\mathrm{Nul}A \subset \mathbb{R}^n$?

(b) Compute the reduced echelon form of

$$A = \begin{bmatrix} 5 & -1 & 6 & -4 \\ 2 & 3 & 16 & -5 \\ 0 & 2 & 8 & -2 \end{bmatrix}.$$

(c) Find a basis for the column space of

$$A = \begin{bmatrix} 5 & -1 & 6 & -4 \\ 2 & 3 & 16 & -5 \\ 0 & 2 & 8 & -2 \end{bmatrix}.$$

(d) Find a basis for the null space of

$$A = \begin{bmatrix} 5 & -1 & 6 & -4 \\ 2 & 3 & 16 & -5 \\ 0 & 2 & 8 & -2 \end{bmatrix}.$$

(e) What are the dimensions of the column space and null space of A?

Problem 5. (5 + 5 + 5 + 5 = 20 points)

(a) Does there exist a 3×3 matrix whose column space contains

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ but not } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}?$$

If there is, give an example. If there isn't, explain why not.

(b) Does there exist a 3×3 matrix whose null space contains

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ but not } \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}?$$

If there is, give an example. If there isn't, explain why not.

(c) Does there exist a 3×3 matrix whose null space *and* column space contains

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}?$$

If there is, give an example. If there isn't, explain why not.

(d) Does there exist a 3×3 matrix whose null space *and* column space contains

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}?$$

If there is, give an example. If there isn't, explain why not.

Problem 6. (10 points) Find all 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $2a$ is a positive integer, b, c, d are real numbers, and $A^T = A^{-1}$.

Hint: for any square matrix B*, recall that* $\det(BB^T) = \det(B)\det(B^T) = \det(B)^2$.