1. Give examples of linear systems in two variables with (a) no solutions, (b) one solutions, (c) infinitely many solutions.

Solution. (a) $\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 1. \end{cases}$ (b) $\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 1. \end{cases}$ (c) $\begin{cases} x_1 + x_2 = 0 \\ 2x_1 + 2x_2 = 0. \end{cases}$

2. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

   (a) Give an example of a linear system whose coefficient matrix is A.

   (b) Give an example of a linear system whose augmented matrix is A.

   (c) Describe all solutions to the system in (b). How many solutions are there?

Solution. (a) $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 4x_3 = 2 \\ 3x_1 + 4x_2 + 5x_3 = 3. \end{cases}$

(b) $\begin{cases} x_1 + 2x_2 = 3 \\ 2x_1 + 3x_2 = 4 \\ 3x_1 + 4x_2 = 5. \end{cases}$

(c) We have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \texttt{RREF}(A).$$

This shows that columns 1 and 2 are the pivot columns of A, so x_1 and x_2 are both basic variables. Therefore the system in (b) has exactly one solution $(x_1, x_2) = (-1, 2)$.

3. Given the definitions of the following: (a) *row operation*, (b) *echelon form*, (c) *reduced echelon form*, (d) *leading entry*, (e) *pivot position*, (f) *pivot column*, (g) *basic variable*, (h) *free variable*.

Solution. See the textbook or lectures notes for definitions.

4. Suppose your phone number is 12345678. Form the 3×3 matrix

$$A = \begin{bmatrix} x & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}.$$

   (You might try this problem with your own phone number instead.)

   (a) Substitute an arbitrary value for x, and then compute the reduced echelon form of A.

   (b) Find another value for x which results in A having a different reduced echelon form.

   (c) Describe the possible values of $\texttt{RREF}(A)$ as a function of x.

Solution. (a) Let's try $x = 0$. Then

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Instead let $x = 3$. Then

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 3 & 1 & 2 \\ 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) The second two columns of A are linearly independent since neither is a scalar multiple of the other. If the first column is not in the span of these two columns, then the reduced echelon form of the matrix will be the 3-by-3 identity matrix as in case (b).

So for what values of x is $\begin{bmatrix} x \\ 3 \\ 6 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$? This is the same as asking for the values of x such that the vector equation

$$y_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} x \\ 3 \\ 6 \end{bmatrix} \qquad (*)$$

has a solution. We solve this vector equation by row reduction:

$$\begin{bmatrix} 1 & 2 & x \\ 4 & 5 & 3 \\ 7 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & x \\ 0 & -3 & 3-4x \\ 0 & -6 & 6-7x \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & x \\ 0 & 3 & -3+4x \\ 0 & -6 & 6-7x \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & x \\ 0 & 3 & -3+4x \\ 0 & 0 & x \end{bmatrix}.$$

The last matrix is only in echelon form, not reduced echelon form. But from this matrix we can already see that the last column will contain a pivot position precisely when $x \neq 0$. The vector equation (*) has no solution if and only if this happens.

Thus if $x \neq 0$ then $\texttt{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and if $x = 0$ then $\texttt{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

5. Given the definitions of (a) *linear combination*, (b) *span*, and (c) *linear independence* of a set of vectors.

Solution. See the textbook or lectures notes for definitions.

6. Determine if the columns of the matrices

$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

are linearly independent.

Solution. The columns of a matrix are linearly independent if every column is a pivot column. In this problem

$$\texttt{RREF}(A) = \texttt{RREF}(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so in both matrices the columns are linearly independent.

7. Compute AB^T and BA^T, with A and B defined as in the previous problem.

Solution. This is just arithmetic. Double check your answer yourself!

8. Do the columns of A or B span $\mathbb{R}^4$?

   Do the columns of A^T or B^T span $\mathbb{R}^3$?

Solution. The columns of A do not span $\mathbb{R}^4$ since A does not have a pivot position in every row (only rows 1, 2, and 3). The same is true for B.

The columns of both A^T and B^T span $\mathbb{R}^3$. The hard but straightforward way to check this is to compute $\texttt{RREF}(A^T)$ and $\texttt{RREF}(B^T)$ and see that there are pivot positions in every row. The easy but slightly tricky way to see this is to note that

$$\texttt{RREF}(A) = \texttt{RREF}(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

implies that there are matrices 4-by-4 matrix E and F such that

$$EA = FB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so by taking transposes we have

$$A^T E^T = B^T F^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now observe that if $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$ is any vector then $A^T x = B^T y = v$ for

$$x = E^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} \qquad \text{and} \qquad y = F^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}.$$

9. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function. Say what it means for f to be (a) *linear*, (b) *one-to-one*, (c) *onto*, (d) *invertible*.

Solution. See the textbook or lectures notes for definitions.

10. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto and linear. What are the possible values for $n - m$?

Solution. If T is onto and linear then $n \geq m$ so the possible values for $n - m$ are $0, 1, 2, 3, 4, 5, \dots$ i.e. any nonnegative integer.

11. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one and linear. What are the possible values for $n - m$?

Solution. If T is one-to-one and linear then $n \leq m$ so the possible values for $n-m$ are $0, -1, -2, -3, -4, -5, \dots$ i.e. any nonpositive integer.

12. Determine if the matrix

$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \end{bmatrix}$$

is invertible. If it is, compute its inverse.

Solution. We can check if A is invertible and compute its inverse at the same time by row reducing

$$\begin{bmatrix} 0 & -8 & 5 & 1 & 0 & 0 \\ 3 & -7 & 4 & 0 & 1 & 0 \\ -1 & 5 & -4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 & 7/24 & -1/8 \\ 0 & 1 & 0 & -1/3 & -5/24 & -5/8 \\ 0 & 0 & 1 & -1/3 & -1/3 & -1 \end{bmatrix}.$$

(I'm not showing my work here, but you should!) Since the first three columns give the identity matrix, $\texttt{RREF}(A) = I_3$ so A is invertible, with inverse

$$A^{-1} = \frac{1}{24} \begin{bmatrix} -8 & 7 & -3 \\ -8 & -5 & -15 \\ -8 & -8 & -24 \end{bmatrix}.$$

13. Consider the matrix

$$A = \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & f & g \\ 0 & 0 & 0 & h & i \end{bmatrix}.$$

What is $\det A$? When is A invertible? Assuming A invertible, given a formula for A^{-1}.

Solution. Observe that $A = BCD$ where

$$B = \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & f & g \\ 0 & 0 & 0 & h & i \end{bmatrix}.$$

The determinant of each of these is easy compute using the recursive rule for determinants:

$$\det B = ad - bc \qquad \text{and} \qquad \det C = e \qquad \text{and} \qquad \det d = fi - gh.$$

Therefore $\det A = (ad - bc)(e)(fi - gh)$.

The matrix A is invertible if $ad - bc$ and e and $fi - gh$ are all nonzero. In this case

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 & 0 & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 & 0 \\ 0 & 0 & 1/e & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{fi-gh} & \frac{-g}{fi-gh} \\ 0 & 0 & 0 & \frac{-h}{fi-gh} & \frac{f}{fi-gh} \end{bmatrix}.$$

14. Given the definition of the following (a) *subspace* of $\mathbb{R}^n$, (b) *basis* of a subspace, (c) *dimension* of a subspace.

Solution. See the textbook or lectures notes for definitions.

15. Let A be an $m \times n$ matrix. Given the definition of the following (a) the *nullspace* of A, (b) the *column space* of A, and (c) the *rank* of A.

Solution. See the textbook or lectures notes for definitions.

16. Suppose A is an $m \times n$ matrix. What are the possible values for $\operatorname{rank} A$? What are the possible values of $\dim \operatorname{Nul} A$?

Solution. We must have $\operatorname{rank} A \in \{0, 1, 2, 3, \dots, \min\{m, n\}\}$ and $\dim \operatorname{Nul} A \in \{0, 1, 2, 3, \dots, n\}$.

Since $\operatorname{Col} A \subset \mathbb{R}^m$ we have $0 \leq \operatorname{rank} A \leq m$, but also $\dim \operatorname{Col} A = n - \dim \operatorname{Nul} A \leq n$.

17. Find a basis for the nullspace of

$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}.$$

Solution. This was not a very interesting matrix to consider. It follows from Problem 12 that

$$\texttt{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the columns of A are linearly independent and $\operatorname{Nul} A = \{0\}$, so the empty set is a basis for $\operatorname{Nul} A$.

18. Find a basis for the column space of A^T, with A defined as in the previous problem.

Solution. We have

$$\texttt{RREF}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -1/4 \end{bmatrix}$$

so the first three columns $\begin{bmatrix} 0 \\ -8 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$ are a basis for $\operatorname{Col} A^T = \mathbb{R}^3$.

A more interesting question would be to ask for a basis of $\operatorname{Nul} A^T$. By the rank theorem, $\operatorname{Nul} A^T$ is 1-dimensional since $\dim \operatorname{Col} A^T + \dim \operatorname{Nul} A^T = 4$. So a basis of $\operatorname{Nul} A$ is given by any nonzero element in the subspace. For example, the vector $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 4 \end{bmatrix}$.

19. Is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \det \begin{bmatrix} 1 & v_1 & 3 \\ 4 & v_2 & 6 \\ 7 & v_3 & 9 \end{bmatrix}$$

a linear transformation $\mathbb{R}^3 \to \mathbb{R}$? If it is, compute its standard matrix.

Solution. Yes it is; this is one of the defining properties of the determinant.

We have

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \det \begin{bmatrix} 1 & v_1 & 3 \\ 4 & v_2 & 6 \\ 7 & v_3 & 9 \end{bmatrix} = (9v_2-6v_3)-v_1(36-42)+3(4v_3-7v_2) = 6v_1-12v_2+6v_3$$

so the standard matrix of T is $A = \begin{bmatrix} 6 & -12 & 6 \end{bmatrix}$.

20. Suppose you have two matrices A and B of the same size. How would you construct a matrix C whose nullspace is the intersection of $\operatorname{Nul} A$ and $\operatorname{Nul} B$?

Solution. Stack the two matrices on top of each other to form $C = \begin{bmatrix} A \\ B \end{bmatrix}$. Then $Cv = \begin{bmatrix} Av \\ Bv \end{bmatrix} = 0$ if and only if $Av = 0$ and $Bv = 0$, so $v \in \operatorname{Nul} C$ if and only if $v \in \operatorname{Nul} A$ and $v \in \operatorname{Nul} B$, so $\operatorname{Nul} C = \operatorname{Nul} A \cap \operatorname{Nul} B$.

21. Suppose you have two matrices A and B of the same size. How would you construct a matrix C whose column space contains both $\operatorname{Col} A$ and $\operatorname{Col} B$?

Just put the two matrices side by side to form $C = \begin{bmatrix} A & B \end{bmatrix}$. The columns of C the include all columns of A and B, and therefore $\operatorname{Col} C$ contains all linear combinations of these columns.

22. Compute the determinant of

$$A = \begin{bmatrix} 0 & x & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix}.$$

Solution. Use the recursive determinant formula to get $\det A = -xyzw$. Alternatively,

$$\det A = xyzw \det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -xyzw$$

since the permutation matrix has inversion number 1.

23. Does there exist a 2×2 matrix A with all entries in $\mathbb{R}$ such that $A^2 v = -v$ for all $v \in \mathbb{R}^2$? If not, say why. If there is, give an example. (Recall that $A^2 = AA$ for a square matrix.)

Solution. If $A^2 v = -v$ for $v \in \mathbb{R}^2$ then multiplication by A^2 acts on the $\mathbb{R}^2$-plane by rotating everything 180 degrees counterclockwise: this reverses the direction of all vectors, sending v to $-v$.

The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ acts on $\mathbb{R}^2$ by rotating all vectors counterclockwise by 90 degrees. Therefore A^2 acts to rotate a given vector 90 degrees twice, i.e., 180 degrees. You can also check directly that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It actually follows from this exercise (with some extra work) that for any $k > 1$, there is a $k \times k$ matrix X with all entries in $\mathbb{R}$ satisfying any polynomial equation of the form

$$a_n X^n + a_{n-1} X^{n-1} + \cdots + a_2 X^2 + a_1 X + a_0 I_k = 0$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. This is not true if $k = 1$, since for example $X^2 + 1 = 0$ has no real solutions $X \in \mathbb{R}$.