TLDR

Quick summary of today's notes. Lecture starts on next page.

Linear independence:

- Vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are *linearly independent* if the only way to express
$$0 = c_1v_1 + c_2v_2 + \cdots + c_pv_p$$
for $c_1, c_2, \dots, c_p \in \mathbb{R}$ is by taking $c_1 = c_2 = \cdots = c_p = 0$. This happens if and only if
$$\{0\} \neq \mathbb{R}\text{-span}\{v_1\} \neq \mathbb{R}\text{-span}\{v_1, v_2\} \neq \mathbb{R}\text{-span}\{v_1, v_2, v_3\} \neq \cdots \neq \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_p\}.$$
- If the vectors are not linearly independent, then they are *linearly dependent*. This happens when
$$\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_{i-1}\} = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_i\}$$
for at least one $i \in \{1, 2, \dots, p\}$. Here we interpret "$\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_{i-1}\}$" to be $\{0\}$ if $i = 1$.
- Two or more vectors are linearly dependent if one of the vectors is in the span of all of the others.
- If $p > n$ then any vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linearly dependent.
- A list of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is linearly dependent if the $n \times p$ matrix
$$A = \begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$$
has at least one column that is not a pivot column.

Functions and linearity:

- Writing $f : X \to Y$ means that f is a function that transforms inputs $x \in X$ to outputs $f(x) \in Y$.
The set X is called the *domain* while Y is called the *codomain* of f.
The *range* of f is the subset $\mathsf{range}(f) = \{f(x) : x \in X\}$ of Y.
- Let m, n be positive integers. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function then the following mean the same thing:
  - For any $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$ it holds that $f(u + v) = f(u) + f(v)$ and $f(c \cdot v) = c \cdot f(v)$.
  - There exists an $m \times n$ matrix A such that $f(v) = Av$ for all $v \in \mathbb{R}^n$.

  Such functions f are said to be *linear*. The matrix A is called the *standard matrix* of f.
- Every linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ has exactly one standard matrix.
- If $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear then its standard matrix is $A = \begin{bmatrix} f(e_1) & f(e_2) & \dots & f(e_n) \end{bmatrix}$ where
$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n, \quad \dots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n.$$

1 Last time: multiplying vectors and matrices

Given a matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and a vector $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ we define

$$\boxed{Av = v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m.}$$

We refer to Av as the product of A and v, or the vector given by multiplying v by A.

Example. We have $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -1+0+3 \\ -5+0+7 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

If A is an $m \times n$ matrix and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $b \in \mathbb{R}^m$, then we call $\boxed{Ax = b}$ a *matrix equation.*

A matrix equation $Ax = b$ has the same solutions as the linear system with augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$.

Theorem. Let A be an $m \times n$ matrix. The following are equivalent:

1. $Ax = b$ has a solution for any $b \in \mathbb{R}^m$.
2. The span of the columns of A is all of $\mathbb{R}^m$.
3. A has a pivot position in every row.

Example. The matrix equation

$$\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

may fail to have a solution since

$$\mathsf{RREF}\left(\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$$

has pivot positions only in rows 1 and 2.

2 Linear independence

We briefly introduced the notion of linear independence last time.

Suppose we have some vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$. Recall that the *span* of a set of vectors is the set of all possible linear combinations that can be formed using the vectors. If you have a smaller set of vectors inside a bigger set, then the span of the smaller set is always contained in the span of the bigger set.

Moreover, if $y = c_1v_1 + c_2v_2 + \cdots + c_pv_p$ for $c_i \in \mathbb{R}$ is any linear combination of our vectors then $\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\} = \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p, y\}$, since if $a_1, \ldots, a_p, b \in \mathbb{R}$ then

$$a_1v_1 + \cdots + a_pv_p + by = (a_1 + bc_1)v_1 + (a_2 + bc_2)v_2 + \cdots + (a_p + bc_p)v_p \in \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\}.$$

Definition. Consider the p spans given by

$$\{0\} \subseteq \mathbb{R}\text{-span}\{v_1\} \subseteq \mathbb{R}\text{-span}\{v_1, v_2\} \subseteq \mathbb{R}\text{-span}\{v_1, v_2, v_3\} \subseteq \cdots \subseteq \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\}.$$

The vectors $v_1, v_2, \ldots, v_p$ are **linearly independent** if these spans are all distinct. That is, if $\mathbb{R}\text{-span}\{v_1\}$ is strictly bigger than the set $\{0\}$ consisting of just the zero vector, and $\mathbb{R}\text{-span}\{v_1, v_2\}$ is strictly bigger than $\mathbb{R}\text{-span}\{v_1\}$, and $\mathbb{R}\text{-span}\{v_1, v_2, v_3\}$ is strictly bigger than $\mathbb{R}\text{-span}\{v_1, v_2\}$, and so on.

Example. If $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then v_1, v_2, v_3 are linearly independent, since

$$\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} \subsetneq \mathbb{R}\text{-span}\{v_1\} = \left\{\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} : a \in \mathbb{R}\right\} \subsetneq \mathbb{R}\text{-span}\{v_1, v_2\} = \left\{\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R}\right\} \subsetneq \mathbb{R}\text{-span}\{v_1, v_2, v_3\} = \left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R}\right\}.$$

Here "$S \subsetneq T$" means "S is contained in T but $S \neq T$."

Example. If $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ then v_1, v_2, v_3 are not linearly independent as

$$\mathbb{R}\text{-span}\{v_1, v_2\} = \mathbb{R}\text{-span}\{v_1, v_2, -v_1 - v_2\} = \mathbb{R}\text{-span}\{v_1, v_2, v_3\}.$$

When vectors are not linearly independent, we say they are **linearly dependent**.

A *linear dependence* among $v_1, v_2, \ldots, v_p$ is a way of writing the zero vector as a linear combination $0 = c_1v_1 + c_2v_2 + \cdots + c_pv_p$ for some scalar coefficients $c_1, c_2, \ldots, c_p \in \mathbb{R}$ that are *not all zero.*

If $0 = c_1v_1 + c_2v_2 + \cdots + c_pv_p$ is a linear dependence then the matrix equation

$$\begin{bmatrix} v_1 & v_2 & \ldots & v_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = 0$$

has two solutions (and therefore infinitely many) given by $(0, 0, \ldots, 0)$ and $(c_1, c_2, \ldots, c_p)$.

Proposition (Another characterization of linear independence)**.** The vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ are linearly independent if and only if no linear dependence exists among them.

Proof. If i is minimal such that there exists a linear dependence $c_1v_1 + c_2v_2 + \cdots + c_iv_i = 0$ then we must have $c_i \neq 0$ (since if $c_i = 0$ then $c_1v_1 + c_2v_2 + \cdots + c_{i-1}v_{i-1} = 0$ would be a shorter dependence). In this case $v_i = -\frac{c_1}{c_i}v_1 - \frac{c_2}{c_i}v_2 - \cdots - \frac{c_{i-1}}{c_i}v_{i-1}$ so $\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_i\}$.

Conversely, if $\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_i\}$ then $v_i \in \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_{i-1}\}$, which means $v_i = a_1v_1 + a_2v_2 + \ldots a_{i-1}v_{i-1}$ some coefficients $a_1, a_2, \ldots, a_{i-1} \in \mathbb{R}$. But then we get a linear dependence $c_1v_1 + c_2v_2 + \cdots + c_iv_i = 0$ by taking $c_1 = a_1$, $c_2 = a_2$, $\ldots$, $c_{i-1} = a_{i-1}$ and $c_i = -1$. ☐

How to determine if $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ are linear independent.

- Form the $n \times p$ matrix $A = \begin{bmatrix} v_1 & v_2 & \ldots & v_p \end{bmatrix}$.
- Reduce A to echelon form to find its pivot columns.
- If every column of A is a pivot column, then the vectors are linearly independent.

  If some column of A is not a pivot column, then the vectors are linearly dependent.

Example. The vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}$ are linear dependent since

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 21 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A)$$

where $\sim$ denotes row equivalence. The last matrix has no pivot position in column 3. In fact, we have

$$-\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0.$$

The vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 9 \\ 15 \end{bmatrix}$ are linearly independent, since

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{RREF}(A).$$

Every column of A contains a pivot position, so the linear system with coefficient matrix A has no free variables, so $Ax = 0$ have no nontrivial solutions, meaning the columns of A are linearly independent.

Facts about linear independence.

1. A single vector v is linearly independent if and only if $v \neq 0$.
2. A list of vectors is linearly dependent if it includes the 0 vector.
3. Vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ are linearly dependent if and only if some vector v_i is a linear combination of the other vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p$.

   We saw this in the previous example: $\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

The following non-obvious fact is often useful for showing that vectors are linearly dependent:

Theorem. Assume $p > n$ and $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$. Then these vectors are linearly dependent.

Proof. Let $A = \begin{bmatrix} v_1 & v_2 & \ldots & v_p \end{bmatrix}$.

This matrix has more columns than rows.

Each row contains at most one pivot position, so there are fewer pivot positions than columns.

Therefore some column is not a pivot column.

This means the linear system $Ax = 0$ has a free variable, so has more than one solution.

This implies that $v_1, v_2, \ldots, v_p$, the columns of A, are linearly dependent. $\square$

Example. The vectors $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 5 \\ 60 \end{bmatrix}$ are linearly dependent since $3 > 2$.

3 Linear transformations

A function f takes an input x from some set X and produces an output $f(x)$ in another set Y.

We write $f : X \to Y$ to mean that f is a function that takes inputs from X and gives outputs in Y.

The set X is called the *domain* of the function f. The set Y is called the *codomain* of f.

For example, the formula $f(x) = \sqrt{x}$ defines a function $X \to Y$ with $X = Y = \{x \in \mathbb{R} : x \geq 0\}$.

The formula $f(x) = |x|$ defines a function $\mathbb{R} \to \mathbb{R}$.

For every x in the domain X of f, we get an output $f(x)$.

It is possible that some values y in the codomain Y may never occur as outputs of f.

The *image* of an input x in X under f is the output $f(x)$. Define the *image* or *range* of the function f to be the subset $\mathsf{range}(f) = \{f(x) : x \in X\}$ of the codomain Y. This is the set of all possible outputs of f.

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function whose domain and codomain are sets of vectors. The function f is a *linear transformation* or a *linear function* if both of these properties hold:

(1) $f(u + v) = f(u) + f(v)$ for all vectors $u, v \in \mathbb{R}^n$.

(2) $f(cv) = cf(v)$ for all vectors $v \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$.

Example. If A is an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ is the function with the formula $T(v) = Av$ for $v \in \mathbb{R}^n$ then T is a linear function.

Linear transformations have some additional properties worth noting:

Proposition. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

(3) $f(0) = 0$.

(4) $f(u - v) = f(u) - f(v)$ for $u, v \in \mathbb{R}^n$.

(5) $f(au + bv) = af(u) + bf(v)$ for all $a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$.

Proof. We have $f(0) = f(0 + 0) = 2f(0)$ so $f(0) = 0$.

We have $f(u - v) = f(u) + f(-v) = f(u) + (-1)f(v) = f(u) - f(v)$.

Finally, we have $f(au + bv) = f(au) + f(bv) = af(u) + bf(v)$. □

Example. Suppose $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function defined by $T(v) = Av$.

(a) The image of a vector $v \in \mathbb{R}^2$ under T is by definition $T(v) = Av$.

The image of $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ under T is $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$.

(b) Is the range of T all of $\mathbb{R}^3$? If it was, then (from results last time) A would have to have a pivot position in every row. This is impossible since each column can only contain one pivot position, but A has three rows and only two columns. Therefore $\mathsf{range}(T) \neq \mathbb{R}^3$.

The fundamental theorem relating matrices and linear transformations:

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then there is a unique $m \times n$ matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^n$.

Moral: **matrices uniquely represent linear transformations $\mathbb{R}^n \to \mathbb{R}^m$.**

Proof. Define $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ as the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so that e_i has a 1 in the ith row and 0 in all other rows.

Define $a_i = T(e_i) \in \mathbb{R}^m$ and $A = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}$. If w is any vector $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$ then

$$T(w) = T(w_1 e_1 + \dots + w_n e_n) = w_1 T(e_1) + \dots + w_n T(e_n) = w_1 a_1 + \dots + w_n a_n = Aw.$$

Thus A is one matrix such that $T(v) = Av$ for all vectors $v \in \mathbb{R}^n$.

To show that A is the only such matrix, suppose B is a $m \times n$ matrix with $T(v) = Bv$ for all $v \in \mathbb{R}^n$.

Then $T(e_i) = Ae_i = Be_i$ for all $i = 1, 2, \dots, n$.

But Ae_i and Be_i are the ith columns of A and B. For example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} e_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Therefore A and B have the same columns, so they are the same matrix: $A = B$. □

We call the matrix A in this theorem the *standard matrix* of the linear transformation T.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function $T(v) = 3v$.

This is a linear transformation. What is the standard matrix A of T?

As we saw in the proof of the theorem, the standard matrix of $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix} = \begin{bmatrix} 3e_1 & 3e_2 & \dots & 3e_n \end{bmatrix} = \begin{bmatrix} 3 & 0 & \dots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{bmatrix}.$$

In words, A is the matrix with 3 in each position $(1,1), (2,2), \dots, (n,n)$ and 0 in all other positions.

One calls such a matrix *diagonal*.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1^2 + v_2^2 + \dots + v_n^2.$$

This function is *not* linear: we have $T(2v) = 4T(v) \neq 2T(v)$ for any nonzero vector $v \in \mathbb{R}^n$.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix}.$$

This function is a linear transformation. (Why?) Its standard matrix is

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_{n-1}) & T(e_n) \end{bmatrix} = \begin{bmatrix} e_n & e_{n-1} & \dots & e_2 & e_1 \end{bmatrix} = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}.$$

In the matrix on the right, we adopt the convention of only writing the nonzero entries: all positions in the matrix which are blank contain zero entries.

4 Vocabulary

Keywords from today's lecture:

1. **Linearly independent** vectors.

   Vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are **linearly independent** if $x_1 v_1 + \cdots + x_p v_p = 0$ holds only if $x_1 = x_2 = \cdots = x_p = 0$; or when $\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$ has a pivot position in every column.

   Vectors that are not linearly independent are **linearly dependent**.

   Example: The three vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ are linearly independent.

   The four vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ are linearly dependent.

2. **Domain** and **codomain** of a function $f : X \to Y$.

   The **domain** X is the set of inputs for the function.

   The **codomain** Y is a set that contains the output of the function. This set can also contain elements that are not outputs of the function.

   Example: If A is an $m \times n$ matrix then the function $T(v) = Av$ has domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$.

3. **Range** of a function $f : X \to Y$.

   The set $\mathsf{range}(f) = \{f(x) : x \in X\} \subset Y$ of all possible outputs of the function f.

   Example: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ has $T(v) = Av$ then $\mathsf{range}(T) = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$.

4. **Linear function** $f : \mathbb{R}^n \to \mathbb{R}^m$.

   A function with $f(cv) = cf(v)$ and $f(u + v) = f(u) + f(v)$ for $c \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$.

   Example: Every such function has the form $f(v) = Av$ for a unique $m \times n$ matrix A.

   The matrix A is called the **standard matrix** of f if $f(v) = Av$ for all $v \in \mathbb{R}^n$.

5. **Diagonal** matrix

   A matrix which has 0 in position (i, j) if $i \neq j$.

   Example: $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$.