TLDR

Quick summary of today's notes. Lecture starts on next page.

- Let H be a subspace of $\mathbb{R}^n$.

  Every basis of H has the same size.

  The size of any basis of H is called its *dimension*. This number is denoted $\dim H$.

  We always have $0 \leq \dim H \leq n$.

  If $\dim H = d$ then we say that H is *d-dimensional*.

  Dimension measures the size of a **subspace**.

  We usually do not think of individual vectors as having dimension, since a single vector belongs to many different subspaces at the same time, all with different dimensions.

- Only the zero subspace has dimension zero.

  The only subspace of $\mathbb{R}^n$ with dimension n is $\mathbb{R}^n$ itself.

  If $U \subset V \subset \mathbb{R}^n$ are subspaces then $0 \leq \dim U \leq \dim V \leq n$.

- If $\mathcal{B} = (v_1, v_2, \dots, v_m)$ is a basis for a subspace H of $\mathbb{R}^n$, then each $h \in H$ can be expressed as

$$h = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \quad \text{for a unique vector} \quad \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \in \mathbb{R}^m.$$

  The vector on the right is the *coordinate vector* of h in the basis $\mathcal{B}$, sometimes denoted $[h]_{\mathcal{B}} \in \mathbb{R}^m$.

- Let A be an $m \times n$ matrix.

  The dimension of $\operatorname{Col} A$ is the number of pivot columns in A.

  The dimension of $\operatorname{Nul} A$ is the number of non-pivot columns in A.

  Consequently $\dim \operatorname{Col} A + \dim \operatorname{Nul} A = n =$ the total number of columns in A.

- The *rank* of A is defined to be $\operatorname{rank} A = \dim \operatorname{Col} A$.

  A is invertible if and only if $\operatorname{rank} A = m = n$.

  Assume $m = n$. Then A is invertible if and only if $\operatorname{Nul} A = \{0\}$.

- Suppose H of $\mathbb{R}^n$ is a subspace and $p = \dim H$.

  Any set of p linearly independent vectors in H is a basis for H.

  Any set of p vectors whose span in H is a basis for H.

1 Last time: inverses and subspaces

To show that an $n \times n$ matrix A is *invertible*, all we have to do is check that (1) its columns are linearly independent or (2) its columns span $\mathbb{R}^n$. If either (1) or (2) holds, then the other property is also true.

If A is invertible then it has an *inverse* which is an $n \times n$ matrix A^{-1} with

$$AA^{-1} = A^{-1}A = I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

If A and B are $n \times n$ and $AB = I_n$ then it automatically holds that $BA = I_n$ so $B = A^{-1}$ and $A = B^{-1}$.

Definition. A subset H of $\mathbb{R}^n$ is a *subspace* if $0 \in H$, $u + v \in H$, and $cv \in H$ for all $u, v \in H$ and $c \in \mathbb{R}$.

A subspace is a nonempty set that contains all linear combinations of vectors already in the set.

Example. Examples of subspaces of $\mathbb{R}^n$:

- The set $\{0\}$ containing just the zero vector.
- The set of all scalar multiples of a single vector.
- $\mathbb{R}^n$ itself.
- The span of any set of vectors in $\mathbb{R}^n$.
- The range of a linear function $T : \mathbb{R}^k \to \mathbb{R}^n$.
- The set of vectors v with $T(v) = 0$ for a linear function $T : \mathbb{R}^n \to \mathbb{R}^k$.

The union of two subspaces is not necessarily a subspace. (Why?)

The intersection of two subspaces is a subspace, however. (Why?)

Definition. To any $m \times n$ matrix A there are two corresponding subspaces of interest:

1. The *column space* of A is the subspace $\operatorname{Col} A$ of $\mathbb{R}^m$ given by the span of the columns of A.
2. The *null space* of A is the subspace $\operatorname{Nul} A$ of $\mathbb{R}^n$ given by the set of vectors $v \in \mathbb{R}^n$ with $Av = 0$.

It is not obvious from these definitions, but it will turn out that each subspace of $\mathbb{R}^m$ occurs as the column space of some matrix. Likewise, each subspace of $\mathbb{R}^n$ occurs as the null space of some matrix.

Definition. A *basis* of a subspace H of $\mathbb{R}^n$ is a set of linearly independent vectors whose span is H.

An important basis with its own notation: the *standard basis* of $\mathbb{R}^n$ consists of the vectors $e_1, e_2, \dots, e_n$ where e_i is the vector in $\mathbb{R}^n$ with 1 in row i and 0 in all other rows.

One fundamental property of subspaces and bases:

Theorem. Every subspace H of $\mathbb{R}^n$ has a basis of size at most n.

Let A be an $m \times n$ matrix.

How to find a basis of $\mathrm{Nul}\,A$**.**

1. Find all solutions to $Ax = 0$ by row reducing A to echelon form. Recall that x_i is a *basic variable* if column i of $\mathsf{RREF}(A)$ contains a leading 1, and that otherwise x_i is a *free variable.*
2. Express each basic variable in terms of the free variables, and then write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_{i_1} b_1 + x_{i_2} b_2 + \dots + x_{i_k} b_k$$

   where $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ are the free variables and $b_1, b_2, \dots, b_k \in \mathbb{R}^n$.
3. The vectors $b_1, b_2, \dots, b_k$ then form a basis for $\mathrm{Nul}\,A$.

Example. Suppose $A = \begin{bmatrix} 1 & 2 & 5 & 8 \\ 2 & 3 & 7 & 0 \end{bmatrix}$.

1. Then $A \sim \begin{bmatrix} 1 & 2 & 5 & 8 \\ 0 & -1 & -3 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -24 \\ 0 & 1 & 3 & 16 \end{bmatrix}$ so $Ax = 0$ iff $\begin{cases} x_1 - x_3 - 24x_4 = 0 \\ x_2 + 3x_3 + 16x_4 = 0. \end{cases}$
2. This means x_1, x_2 are basic variables and x_3, x_4 are free variables.

   We have $Ax = 0$ if and only if $x_1 = x_3 + 24x_4$ and $x_2 = -3x_3 - 16x_4$, in which case

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 24x_4 \\ -3x_3 - 16x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 24 \\ -16 \\ 0 \\ 1 \end{bmatrix}.$$

3. The set of vectors $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 24 \\ -16 \\ 0 \\ 1 \end{bmatrix} \right\}$ is then a basis for $\mathrm{Nul}\,A$.

How to find a basis of $\mathrm{Col}\,A$**.**

1. The pivot columns of A form a basis of $\mathrm{Col}\,A$.

This looks simpler than the previous algorithm, but to find out which columns of A are pivot columns, we have to row reduce A to echelon form, which takes just as much work as finding a basis of $\mathrm{Nul}\,A$.

Example. If $A = \begin{bmatrix} 1 & 2 & 5 & 8 \\ 2 & 3 & 7 & 0 \end{bmatrix}$ then columns 1, 2 have pivots so $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}\,A$.

This is not the only set of columns of A that forms a basis for $\mathrm{Col}\,A$, however.

2 Coordinate systems

Suppose H is a subspace of $\mathbb{R}^n$. Let $b_1, b_2, \dots, b_k$ be a basis of H.

Theorem. Let $v \in H$. There are unique coefficients $c_1, c_2, \dots, c_k \in \mathbb{R}$ such that

$$c_1 b_1 + c_2 b_2 + \dots + c_k b_k = v.$$

Proof. Since our basis spans H, there must be some coefficients with $c_1b_1 + c_2b_2 + \cdots + c_kb_k = v$. If these coefficients were not unique, so that we could write $c'_1b_1 + c'_2b_2 + \cdots + c'_kb_k = v$ for some different list of numbers $c'_1, c'_2, \dots, c'_k \in \mathbb{R}$, then we would have

$$\begin{aligned} 0 = v - v &= (c_1b_1 + c_2b_2 + \cdots + c_kb_k) - (c'_1b_1 + c'_2b_2 + \cdots + c'_kb_k) \\ &= (c_1 - c'_1)b_1 + (c_2 - c'_2)b_2 + \cdots + (c_k - c'_k)b_k. \end{aligned}$$

In this case, since our numbers are different, at least one of the differences $c_i - c'_i$ must be nonzero, and so what we just wrote is a nontrivial linear dependence among the vectors $b_1, b_2, \dots, b_k$. But this is impossible since the elements of a basis are linearly independent. □

Let $\mathcal{B} = (b_1, b_2, \dots, b_k)$ be the list consisting of our basis vectors in some fixed order.

Given $v \in H$, define $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$ as the unique vector with $c_1b_1 + c_2b_2 + \cdots + c_kv_k = v$.

Equivalently, $[v]_{\mathcal{B}}$ is the unique solution to the matrix equation $\begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} x = v$.

We call $[v]_{\mathcal{B}}$ the *coordinate vector of* v *in the basis* $\mathcal{B}$ or just v *in the basis* $\mathcal{B}$.

Example. If $H = \mathbb{R}^n$ and $\mathcal{B} = (e_1, e_2, \dots, e_n)$ is the standard basis then $[v]_{\mathcal{B}} = v$.

Example. If $H = \mathbb{R}^n$ and $\mathcal{B} = (e_n, \dots, e_2, e_1)$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then $[v]_{\mathcal{B}} = \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix}$.

Example. Let $b_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$.

Then $\mathcal{B} = (b_1, b_2)$ is a basis for $H = \mathbb{R}\text{-span}\{b_1, b_2\}$, which is a subspace of $\mathbb{R}^3$.

The unique $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ such that $\begin{bmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ is found by row reduction:

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last matrix implies that $x_1 = 2$ and $x_2 = 3$ so $[v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Example. If $b_1 = e_1 - e_2$, $b_2 = e_2 - e_3$, $b_3 = e_3 - e_4$, $\dots$, $b_{n-1} = e_{n-1} - e_n$ and

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ -v_1 - v_2 - \cdots - v_{n-1} \end{bmatrix}$$

then $v \in H = \mathbb{R}\text{-span}\{b_1, b_2, \dots, b_{n-1}\}$ and

$$[v]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_1 + v_2 \\ v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 + v_4 \\ \vdots \\ v_1 + v_2 + v_3 + \dots + v_{n-1} \end{bmatrix} \in \mathbb{R}^{n-1}.$$

The notation $[v]_{\mathcal{B}}$ gives us an easy way to check the following important property:

Theorem. Let H be a subspace of $\mathbb{R}^n$. Then all bases of H have the same number of elements.

Proof. Suppose $\mathcal{B} = (b_1, b_2, \dots, b_k)$ and $\mathcal{B}' = (b'_1, b'_2, \dots, b'_l)$ are two (ordered) bases of H with $k < l$.

Then $[b'_1]_{\mathcal{B}}$, $[b'_2]_{\mathcal{B}}$, …, $[b'_l]_{\mathcal{B}}$ are $l > k$ vectors in $\mathbb{R}^k$, so they must be linearly dependent.

This means there exist coefficients $c_1, c_2, \dots, c_l \in \mathbb{R}$, not all zero, with

$$c_1[b'_1]_{\mathcal{B}} + c_2[b'_2]_{\mathcal{B}} + \dots + c_l[b'_l]_{\mathcal{B}} = 0.$$

But we have $c_1[b'_1]_{\mathcal{B}} + c_2[b'_2]_{\mathcal{B}} + \dots + c_l[b'_l]_{\mathcal{B}} = [c_1b'_1 + c_2b'_2 + \dots + c_lb'_l]_{\mathcal{B}}$.

(This is the key step; why is this true? Think about how $[v]_{\mathcal{B}}$ is defined.)

Thus $[c_1b'_1 + c_2b'_2 + \dots + c_lb'_l]_{\mathcal{B}} = 0$, so

$$c_1b'_1 + c_2b'_2 + \dots + c_lb'_l = \begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} [c_1b'_1 + c_2b'_2 + \dots + c_lb'_l]_{\mathcal{B}} = 0.$$

(The first equality holds since by definition $v = \begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} [v]_{\mathcal{B}}$.)

Since the coefficients c_i are not all zero, this contradicts the fact that $b'_1, b'_2, \dots, b'_l$ are linearly independent. This means our original assumption that H has two bases of different sizes is impossible. □

3 Dimension

Let $\mathcal{B} = (b_1, b_2, \dots, b_k)$ be an ordered basis of a subspace H of $\mathbb{R}^n$.

The function $H \to \mathbb{R}^k$ with the formula $v \mapsto [v]_{\mathcal{B}}$ is linear and invertible.

Thus H "looks the same as" $\mathbb{R}^k$.

For this reason we say that H is *k-dimensional.* More generally:

Definition. The *dimension* of a subspace H is the number of vectors in any basis of H.

We denote the dimension of H by $\dim H$. This number belongs to $\{0, 1, 2, 3, \dots\}$.

If $H = \{0\}$ then we define $\dim H = 0$.

Example. We have $\dim \mathbb{R}^n = n$.

If H is the set of all vectors of the form $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$, then H is a subspace and $\dim H = k$.

Note that $e_1, e_2, \dots, e_k$ is a basis for H.

A line in $\mathbb{R}^2$ through the origin is a 1-dimensional subspace.

Let A be an $m \times n$ matrix.

The processes we gave to construct bases of $\operatorname{Nul} A$ and $\operatorname{Col} A$ imply that:

Corollary. The dimension of $\operatorname{Nul} A$ is the number of free variables in the linear system $Ax = 0$.

Corollary. The dimension of $\operatorname{Col} A$ is the number of pivot columns in A.

There is a special name for the dimension of the column space of a matrix:

Definition. The *rank* of a matrix A is $\operatorname{rank} A = \dim \operatorname{Col} A$.

Putting everything together gives the following pair of important results.

Theorem (Rank-nullity theorem)**.** If A is a matrix with n columns then $\operatorname{rank} A + \dim \operatorname{Nul} A = n$.

Proof. The number of free variables in the system $Ax = 0$ is also the number non-pivot columns in A.

Therefore $\operatorname{rank} A + \dim \operatorname{Nul} A$ is the total number of columns in A. □

Theorem (Basis theorem)**.** If H is a subspace of $\mathbb{R}^n$ with $\dim H = p$ then

1. Any set of p linearly independent vectors in H is a basis for H.
2. Any set of p vectors in H whose span is H is a basis for H.

Proof. Suppose we have p linearly independent vectors in H. If these vectors do not span H, then adding a vector which is in H but not in their span gives a set of $p + 1$ linearly independent vectors in H.

If this larger set still does not span H, then adding a vector from H that is not in the span gives an even larger linearly independent set of $p + 2$ vectors.

Continuing in this way must eventually produce a basis for H, but this basis will have more than p elements, contradicting $\dim H = p$.

Suppose we instead have p vectors whose span is H. If these vectors are linearly dependent, then one of the vectors is a linear combination of the others. Remove this vector to get $p - 1$ vectors that span H.

If these vectors are still not linearly independent, then one is a linear combination of the others and removing this vector gives a set of $p - 2$ vectors that span H.

Continuing in this way must eventually produce a basis for H, but this basis will have fewer than p elements, contradicting $\dim H = p$. □

Corollary. If H is an n-dimensional subspace of $\mathbb{R}^n$ then $H = \mathbb{R}^n$.

Proof. If H has a basis with n elements then these elements are linearly independent, so form a basis for $\mathbb{R}^n$. Then every vector in $\mathbb{R}^n$ is a linear combination of the basis vectors, so belongs to H. □

If U and V are two sets then we write "$U \subset V$" or "$U \subseteq V$" to mean that every element of U is also an element of V. Both notations mean the same thing. If $U \subset V$ then it could be true that $U = V$.

Sometimes people write "$U \subsetneq V$" to mean "$U \subset V$ but $U \neq V$."

It holds that $U = V$ if and only if we have both $U \subset V$ and $V \subset U$.

Corollary. If $U, V \subset \mathbb{R}^n$ are subspaces with $U \subset V$ but $U \neq V$, then $\dim U < \dim V \leq n$.

Proof. If $j = \dim V \leq \dim U = k$ and $u_1, u_2, \dots, u_k$ is a basis for U, then $u_1, u_2, \dots, u_j$ would be linearly independent and therefore a basis for V. But then $V \subset U$ which would imply $U = V$ if also $U \subset V$. □

Corollary. Let A be an $n \times n$ matrix. The following are equivalent:

(a) A is invertible.

(b) The columns of A form a basis for $\mathbb{R}^n$.

(c) $\operatorname{rank} A = \dim \operatorname{Col} A = n$.

(d) $\dim \operatorname{Nul} A = 0$.

Proof. We have already seen that (a) and (b) are equivalent.

(c) holds if and only if the columns of A span $\mathbb{R}^n$ which is equivalent to (a).

(d) holds if and only if the columns of A are linearly independent which is equivalent to (a). □

4 Vocabulary

Keywords from today's lecture:

1. **Coordinate vector** of a vector $v \in H$ with respect to an ordered basis $\mathcal{B} = (b_1, b_2, \dots, b_k)$.

   The unique vector of coefficients $[v]_\mathcal{B} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$ with $c_1 b_1 + c_2 b_2 + \dots + c_k b_k = v$.

   Example: If $H = \mathbb{R}^2$ and $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ and $v = \begin{bmatrix} x \\ y \end{bmatrix}$ then $[v]_\mathcal{B} = \begin{bmatrix} x - y \\ y \end{bmatrix}$.

2. **Dimension** of a subspace $H \subset \mathbb{R}^n$

   The number $\dim H$ of vectors in any basis for H.

3. **Rank** of an $m \times n$ matrix A.

   The dimension of the column space $\operatorname{Col} A$. This is also the number of pivot columns in A.

   This is denoted $\operatorname{rank} A$.

4. **Rank-nullity theorem**.

   If A is an $m \times n$ matrix then $\dim \operatorname{Col} A + \dim \operatorname{Nul} A = \operatorname{rank} A + \dim \operatorname{Nul} A = n$.

5. **Basis theorem**.

   If $H \subset \mathbb{R}^n$ is a subspace with $\dim H = p$ then (1) any set of p linearly independent vectors in H is a basis for H and (2) any set of p vectors whose span is H is a basis for H.