TLDR

Quick summary of today's notes. Lecture starts on next page.

- Given real numbers $a, b \in \mathbb{R}$, define $a + bi = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

  In this notation, we think of 1 as the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and i as the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

  The set of *complex numbers* is $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.

  We view $\mathbb{R}$ as a subset of $\mathbb{C}$ by setting $a = a + 0i$.

- We can add, subtract, multiply, and take inverses of complex numbers, since they are 2×2 matrices.

  The set of $\mathbb{C}$ is closed under these operations.

  The identity "$i^2 = -1$" holds in the sense that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

- Once we get used to these operations, another useful way to view the elements of $\mathbb{C}$ is as formal expressions $a + bi$ where $a, b \in \mathbb{R}$ and i is a symbol that satisfies $i^2 = -1$.

  Addition, subtraction, and multiplication works just like polynomials, but substituting -1 for i^2.

- Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0$ is a polynomial with coefficients $a_0, a_1, \dots, a_n \in \mathbb{C}$.

  Assume $a_n \neq 0$ so that $p(x)$ has *degree* n.

  Then there are are n (not necessarily distinct) complex numbers $r_1, r_2, \dots, r_n \in \mathbb{C}$ such that

  $$p(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n).$$

  The numbers $r_1, r_2, \dots, r_n$ are the *roots* of $p(x)$.

- The characteristic equation of an $n \times n$ matrix A is a degree n polynomial with real coefficients.

  Counting multiplicities, $\det(A - xI)$ has exactly n roots but some roots may be complex numbers.

- Define $\mathbb{C}^n$ to be the set of vectors $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ with n rows and entries $v_1, v_2, \dots, v_n \in \mathbb{C}$.

  We have $\mathbb{R}^n \subset \mathbb{C}^n$ since $\mathbb{R} = \{a \in \mathbb{R}\} = \{a + 0i : a \in \mathbb{R}\} \subset \mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.

- The sum $u + v$ and scalar multiple cv for $u, v \in \mathbb{C}^n$ and $c \in \mathbb{C}$ are defined exactly as for vectors in $\mathbb{R}^n$, except we use the addition and multiplication operations from $\mathbb{C}$ instead of $\mathbb{R}$.

- If A is an $n \times n$ matrix and $v \in \mathbb{C}^n$ then we define Av in the same way as when $v \in \mathbb{R}^n$.

  Let A be an $n \times n$ matrix whose entries are all real numbers.

  Call $\lambda \in \mathbb{C}$ a *(complex) eigenvalue* of A if there exists a nonzero vector $v \in \mathbb{C}^n$ such that $Av = \lambda v$.

  Equivalently, $\lambda \in \mathbb{C}$ is an eigenvalue of A if λ is a root of the characteristic polynomial $\det(A - xI)$.

  This is no different from our first definition of an eigenvalue, except that now we permit $\lambda \in \mathbb{C}$.

1 Last time: methods to check diagonalizability

Let n be a positive integer and let A be an $n \times n$ matrix.

Remember that A is *diagonalizable* if $A = PDP^{-1}$ where P is an invertible $n \times n$ matrix and D is an $n \times n$ diagonal matrix. In other words, A is diagonalizable if A is similar to a diagonal matrix.

Suppose $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ are linearly independent vectors and $\lambda_1, \lambda_2, \dots, \lambda_n$ are numbers. Define

$$P = \left[\begin{array}{cccc} v_1 & v_2 & \dots & v_n \end{array}\right] \qquad \text{and} \qquad D = \left[\begin{array}{cccc} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{array}\right].$$

If $A = PDP^{-1}$ then $Av_i = PDP^{-1}v_i = PDe_i = \lambda_i Pe_i = \lambda_i v_i$ for each $i = 1, 2, \dots, n$.

In other words, when $A = PDP^{-1}$, the columns of P are a basis for $\mathbb{R}^n$ made up of eigenvectors of A.

Matrices that are not diagonalizable.

Proposition. $\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$ is not diagonalizable.

Proof. To check this directly, suppose $ad - bc \neq 0$ and compute

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right] = \frac{1}{ad-bc}\left[\begin{array}{cc} -ac & a^2 \\ -c^2 & ac \end{array}\right].$$

The only way the last matrix can be diagonal is if $a = c = 0$, but then we would have $ad - bc = 0$ so $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ would not be invertible. Therefore $\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$ is not similar to a diagonal matrix. □

Here is a second family of examples.

Let A be an $n \times n$ upper-triangular matrix with all entries on the diagonal equal to 1:

$$A = \left[\begin{array}{cccc} 1 & * & \dots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & * \\ & & & 1 \end{array}\right]$$

All entries in A below the diagonal are zero, and the entries above the diagonal can be anything.

Proposition. If $A \neq I$ is not the identity matrix then A is not diagonalizable.

Proof. Suppose $A = PDP^{-1}$ where D is diagonal.

Every diagonal entry of D is an eigenvalue for A. (Why?)

But A has characteristic polynomial $(1 - x)^n$ so its only eigenvalue is 1.

Therefore $D = I$ so $A = PIP^{-1} = PP^{-1} = I$. □

The following result summarizes everything we need to know about diagonalizability: how to determine if a matrix A is diagonalizable, and then how to compute the decomposition $A = PDP^{-1}$ if it exists.

Theorem. Let A be an $n \times n$ matrix.

Suppose $\lambda_1, \lambda_2, \dots, \lambda_p$ are the distinct eigenvalues of A.

Let $d_i = \dim \mathrm{Nul}(A - \lambda_i I)$ for $i = 1, 2, \dots, p$.

By the definition of an eigenvalue, we have $1 \le d_i \le n$ for each i. Moreover, the following holds:

1. It holds that $d_1 + d_2 + \cdots + d_p \le n$.
2. The matrix A is diagonalizable if and only if $d_1 + d_2 + \cdots + d_p = n$.
3. Suppose A is diagonalizable. Let $D_i = \lambda_i I_{d_i}$ and define D as the $n \times n$ diagonal matrix

$$D = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_p \end{bmatrix}.$$

Choose n vectors $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ such that the first d_1 vectors are a basis for $\mathrm{Nul}(A - \lambda_1 I)$, the next d_2 vectors are a basis for $\mathrm{Nul}(A - \lambda_2 I)$, the next d_3 vectors are a basis for $\mathrm{Nul}(A - \lambda_3 I)$, and so on, so that the last d_p vectors are basis for $\mathrm{Nul}(A - \lambda_p I)$. Then $A = PDP^{-1}$ for

$$P = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}.$$

2 Complex numbers

For the rest of this lecture, let $i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Recall that $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Suppose $a, b \in \mathbb{R}$. Both i and I_2 are 2×2 matrices, so we can form the sum $aI_2 + bi$.

To simplify our notation, we will write 1 instead of I_2 and $a + bi$ instead of $aI_2 + bi$.

We consider $a = a + 0i$ and $bi = 0 + bi$ and $0 = 0 + 0i$. With this convention, we have

$$a + bi = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Define $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$. This is called the set of *complex numbers*.

According to our definition, each element of $\mathbb{C}$ is a 2×2 matrix, to be called a *complex number*.

Fact. We can add complex numbers together. If $a, b, c, d \in \mathbb{R}$ then

$$(a + bi) + (c + di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a + c & -b - d \\ b + d & a + c \end{bmatrix} = (a + c) + (b + d)i \in \mathbb{C}.$$

Clearly $\boxed{(a + bi) + (c + di) = (c + di) + (a + bi) = (a + c) + (b + d)i}$.

Fact. We can subtract complex numbers. If $a, b, c, d \in \mathbb{R}$ then

$$(a + bi) - (c + di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} - \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a - c & -b + d \\ b - d & a - c \end{bmatrix} = (a - c) + (b - d)i \in \mathbb{C}.$$

Fact. We can multiply complex numbers. If $a, b, c, d \in \mathbb{R}$ then

$$(a+bi)(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix} = (ac-bd)+(ad+bc)i \in \mathbb{C}.$$

Note that $\boxed{(a+bi)(c+di) = (c+di)(a+bi) = (ac-bd)+(ad+bc)i}$.

Fact. We can multiply complex numbers by real numbers. If $a, b, x \in \mathbb{R}$ then define

$$(a+bi)x = x(a+bi) = x\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} ax & -bx \\ bx & ax \end{bmatrix} = (ax)+(bx)i \in \mathbb{C}.$$

Note that this is the same as the product $(a+bi)(x+0i)$.

Fact. We can divide complex numbers by nonzero real numbers. If $a, b, x \in \mathbb{R}$ and $x \neq 0$ then define

$$(a+bi)/x = (a+bi)(1/x) = (a/x)+(b/x)i.$$

We sometimes write $\frac{p}{q}$ instead of p/q. Both expressions means the same thing.

A complex number $a+bi$ is *nonzero* if $a \neq 0$ or $b \neq 0$. Since

$$\det(a+bi) = \det\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a^2+b^2,$$

which is only zero if $a = b = 0$, every nonzero complex number is invertible as a matrix.

Fact. This fact lets us divide complex numbers. If $a, b, c, d \in \mathbb{R}$ and $c+di \neq 0$ then define

$$(a+bi)/(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1}.$$

We can write this more explicitly as

$$\begin{aligned} (a+bi)/(c+di) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} \\ &= \frac{1}{c^2+d^2}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \\ &= \frac{1}{c^2+d^2}\begin{bmatrix} ac+bd & bc-ad \\ ad-bc & ac+bd \end{bmatrix} = \frac{ac+bd}{c^2+d^2} + \frac{ad-bc}{c^2+d^2}i \in \mathbb{C}. \end{aligned}$$

The last formula, while very explicit, is not so easy to remember.

It may be easier to divide complex numbers using the following method:

Example. We have $\frac{3-4i}{2+i} = \frac{(3-4i)(2-i)}{(2+i)(2-i)} = \frac{6-3i-8i+4i^2}{4-i^2} = \frac{6-11i-4}{5} = \frac{2-11i}{5} = \frac{2}{5} - \frac{11}{5}i.$

More generally, if $c+di \neq 0$ then we always have $\boxed{\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2}}$ since

$$\frac{a+bi}{c+di} = (a+bi)(c+di)^{-1} = \frac{1}{c^2+d^2}(a+bi)(c-di) = \frac{(a+bi)(c-di)}{c^2+d^2}.$$

The *complex conjugate* of $c+di$ is defined to be the complex number

$$\overline{c+di} = (c+di)^T = c-di \in \mathbb{C}.$$

When $c + di$ is nonzero, the complex conjugate is related to the inverse by the identity

$$(c+di)^{-1} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \frac{1}{c^2+d^2}\begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \frac{1}{c^2+d^2}\cdot\overline{c+di}.$$

Since $x, y \in \mathbb{C}$ satisfy $xy = yx$ and $(xy)^T = y^T x^T$ (since complex numbers are matrices), it follows that

$$\boxed{\overline{xy} = \overline{y}\cdot\overline{x} = \overline{x}\cdot\overline{y}.}$$

We can also add complex numbers $a + bi$ with real numbers c when $a, b, c \in \mathbb{R}$.

To do this, we set $c = c + 0i$ and define $(a + bi) + c = c + (a + bi) = (a + bi) + (c + 0i) = (a + c) + bi$.

Under this convention, we have

$$\begin{aligned} i^2+1 = (0+i)(0+i)+(1+0i) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0+0i = 0. \end{aligned}$$

Thus it makes sense to write $\boxed{i^2 = -1}$. In a similar way:

Theorem. Define the exponential function $\mathbb{C} \to \mathbb{C}$ by the convergent power series

$$e^x = 1 + \frac{1}{1}x + \frac{1}{1\cdot 2}x^2 + \frac{1}{1\cdot 2\cdot 3}x^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4}x^4 + \dots$$

Then $e^1 = e = 2.71828\dots$ and $\boxed{e^{i\pi} + 1 = 0}$.

Proof. We need two facts from calculus:

$$\begin{aligned} -1 = \cos\pi &= 1 - \frac{1}{1\cdot 2}\pi^2 + \frac{1}{1\cdot 2\cdot 3\cdot 4}\pi^4 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\pi^6 + \dots \\ 0 = \sin\pi &= \frac{1}{1}\pi - \frac{1}{1\cdot 2\cdot 3}\pi^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^5 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\pi^7 + \dots \end{aligned}$$

We have

$$i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad i^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad i^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad i^0 = i^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $i^{n+4} = i^n$ for all n.

Also, we have $(i\pi)^n = \pi^n i^n$. It follows that

$$e^{i\pi} = \begin{bmatrix} 1 - \frac{1}{1\cdot 2}\pi^2 + \frac{1}{1\cdot 2\cdot 3\cdot 4}\pi^4 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\pi^6 + \dots & \frac{1}{1}\pi - \frac{1}{1\cdot 2\cdot 3}\pi^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^5 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\pi^7 + \dots \\ \frac{1}{1}\pi - \frac{1}{1\cdot 2\cdot 3}\pi^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^5 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\pi^7 + \dots & 1 - \frac{1}{1\cdot 2}\pi^2 + \frac{1}{1\cdot 2\cdot 3\cdot 4}\pi^4 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\pi^6 + \dots \end{bmatrix}.$$

By our two facts, this is just $e^{i\pi} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1 + 0i$. Thus $e^{i\pi} + 1 = (-1 + 0i) + (1 + 0i) = 0$. □

After a while, we tend to forget that complex numbers are 2×2 matrices and instead view the elements of $\mathbb{C}$ as formal expressions $a + bi$ where $a, b \in \mathbb{R}$ and i is a symbol that satisfies $i^2 = -1$.

We can add, subtract, and multiply such expressions just like polynomials, but substituting -1 for i^2. This convention gives the same operations as we saw above.

Moreover, this makes it clearer how to view $\mathbb{R}$ as a subset of $\mathbb{C}$, by setting $a = a + 0i$.

The *real part* of a complex number $a + bi \in \mathbb{C}$ is $\mathsf{Re}(a + bi) = a \in \mathbb{R}$.

The *imaginary part* of $a + bi \in \mathbb{C}$ is $\mathsf{Im}(a + bi) = b \in \mathbb{R}$.

Remark. It can be helpful to draw the complex number $a + bi \in \mathbb{C}$ as the vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$.

The number $i(a + bi) = -b + ai \in \mathbb{C}$ then corresponds to the vector $\begin{bmatrix} -b \\ a \end{bmatrix} \in \mathbb{R}^2$, which is given by *rotating* $\begin{bmatrix} a \\ b \end{bmatrix}$ *ninety degrees counterclockwise.* (Try drawing this yourself.)

The main reason it is helpful to work with complex numbers is the following theorem about polynomials.

Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ is a polynomial with coefficients $a_0, a_1, \dots, a_n \in \mathbb{C}$.

Assume $a_n \neq 0$ so that $p(x)$ has *degree* n.

Even though we think of complex numbers are 2×2 matrices, this expression for $p(x)$ still makes sense for $x \in \mathbb{C}$: if we plug in any complex number for x then $a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ is a complex number.

Theorem (Fundamental theorem of algebra)**.** Define $p(x)$ as above. There are n (not necessarily distinct) complex numbers $r_1, r_2, \dots, r_n \in \mathbb{C}$ such that $p(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$.

One calls the numbers $r_1, r_2, \dots, r_n$ the *roots* of $p(x)$.

A root r has *multiplicity* m if exactly m of the numbers $r_1, r_2, \dots, r_n$ are equal to r.

The use of complex numbers in this theorem is essential. The statement fails if we use $\mathbb{R}$ instead of $\mathbb{C}$.

Example: if $p(x) = x^2 + 1$ then there **do not exist** real numbers $r_1, r_2 \in \mathbb{R}$ with $p(x) = (x - r_1)(x - r_2)$.

However, we do have $x^2 + 1 = (x - i)(x + i)$.

3 Complex eigenvalues

The characteristic equation of an $n \times n$ matrix A is a degree n polynomial with real coefficients.

Counting multiplicities, $\det(A - xI)$ has exactly n roots but some roots may be complex numbers.

Define $\mathbb{C}^n$ to be the set of vectors $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ with n rows and entries $v_1, v_2, \dots, v_n \in \mathbb{C}$.

Note that $\mathbb{R}^n \subset \mathbb{C}^n$ since $\mathbb{R} = \{a \in \mathbb{R}\} \subset \mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.

The sum $u + v$ and scalar multiple cv for $u, v \in \mathbb{C}^n$ and $c \in \mathbb{C}$ are defined exactly as for vectors in $\mathbb{R}^n$, except we use the addition and multiplication operations from $\mathbb{C}$ instead of $\mathbb{R}$.

If A is an $n \times n$ matrix and $v \in \mathbb{C}^n$ then we define Av in the same way as when $v \in \mathbb{R}^n$.

Definition. Let A be an $n \times n$ matrix whose entries are all real numbers. Call $\lambda \in \mathbb{C}$ a *(complex) eigenvalue* of A if there exists a nonzero vector $v \in \mathbb{C}^n$ such that $Av = \lambda v$.

Equivalently, $\lambda \in \mathbb{C}$ is an eigenvalue of A if λ is a root of the characteristic polynomial $\det(A - xI)$.

This is no different from our first definition of an eigenvalue, except that now we permit λ to be in $\mathbb{C}$.

Example. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then $\det(A - xI) = x^2 + 1 = (i - x)(-i - x)$.

The roots of this polynomial are the complex numbers i and $-i$. We have

$$A \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \text{and} \qquad A \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

so i and $-i$ are eigenvalues of A, with corresponding eigenvectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Example. Let $A = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix}$. Then $\det(A - xI) = \det \begin{bmatrix} .5 - x & -.6 \\ .75 & 1.1 - x \end{bmatrix} = x^2 - 1.6x + 1$.

Via the quadratic formula, we find that the roots of this characteristic polynomial are

$$x = \frac{1.6 \pm \sqrt{1.6^2 - 4}}{2} = .8 \pm .6i$$

since $i = \sqrt{-1}$. To find a basis for the $(.8 - .6i)$-eigenspace, we row reduce as usual

$$\begin{aligned} A - (.8 - .6i)I &= \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix} - \begin{bmatrix} .8 - .6i & 0 \\ 0 & .8 - .6i \end{bmatrix} = \begin{bmatrix} -.3 + .6i & -.6 \\ .75 & .3 + .6i \end{bmatrix} \\ &\sim \begin{bmatrix} .5 - i & 1 \\ 1 & .8(.5 + i) \end{bmatrix} \sim \begin{bmatrix} 1 & .8(.5 + i) \\ .5 - i & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & .8(.5 + i) \\ 0 & 1 - .8(.5 + i)(.5 - i) \end{bmatrix} = \begin{bmatrix} 1 & .8(.5 + i) \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The last equality holds since $.8(.5 + i)(.5 - i) = .8(.25 - i^2) = .8(1.25) = 1$.

This implies that $Ax = (.8 - .6i)x$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 + .8(.5 + i)x_2 = 0$, i.e., where $5x_1 = -4(.5 + i)x_2 = -(2 + 4i)x_2$. Satisfying these conditions is the vector

$$v = \begin{bmatrix} -2 - 4i \\ 5 \end{bmatrix}$$

which is therefore an eigenvector for A with eigenvalue $.8 - .6i$.

Similar calculations show that the vector $w = \begin{bmatrix} -2 + 4i \\ 5 \end{bmatrix}$ is an eigenvector for A with eigenvalue $.8 + .6i$.

Proposition. Suppose A is an $n \times n$ matrix with real entries. If A has a complex eigenvalue $\lambda \in \mathbb{C}$ with eigenvector $v \in \mathbb{C}^n$ then $\overline{v} \in \mathbb{C}^n$ is an eigenvector for A with eigenvalue $\overline{\lambda}$.

Proof. Since A has real entries, it holds that $\overline{A} = A$. Therefore $A\overline{v} = \overline{A}\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\overline{v}$. □

4 Vocabulary

Keywords from today's lecture:

1. **Complex number**.

   We define a complex number to be either

   - A matrix $a + bi = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a, b \in \mathbb{R}$ and $i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
   - A formal expression "$a + bi$" where $a, b \in \mathbb{R}$ and i is a symbol that has $i^2 = -1$.

   The first definition makes it clear how to add, subtract, multiply, and divide complex numbers (use matrix operations). The second definition is secretly just a way of abbreviating the first definition.

   The set of complex numbers is denoted $\mathbb{C}$.

   Example:

   $1 + 2i$ corresponds to $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

   $(1 + 2i) + (2 + 3i) = 3 + 5i$ corresponds to $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}$.

   $(1 + 2i)(2 + 3i) = -4 + 7i$ corresponds to $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ 7 & -4 \end{bmatrix}$.

   $(1 + 2i)^{-1} = \frac{1}{5} - \frac{2}{5}i$ corresponds to $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

2. **Complex conjugation**.

   If $a, b \in \mathbb{R}$ then *complex conjugate* of $a + bi \in \mathbb{C}$ is $\overline{a + bi} = a - bi \in \mathbb{C}$.

   If $y, z \in \mathbb{C}$ then $\overline{y + z} = \overline{y} + \overline{z}$ and $\overline{yz} = \overline{y} \cdot \overline{z}$ and $\overline{y^{-1}} = \overline{y}^{-1}$.

3. **Fundamental theorem of algebra**.

   Any polynomial
   $$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
   with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$ can be factored as
   $$f(x) = a_n(x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$
   for some not necessarily distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$.

4. **(Complex) eigenvalues and eigenvectors**.

   Let $\mathbb{C}^n$ be the set of vectors with n rows with entries in $\mathbb{C}$. Since $\mathbb{R} \subset \mathbb{C}$, we have $\mathbb{R}^n \subset \mathbb{C}^n$.

   If A is an $n \times n$ matrix and there exists a nonzero vector $v \in \mathbb{C}^n$ with $Av = \lambda v$ for some $\lambda \in \mathbb{C}$, then λ is an *eigenvalue* for A. The vector v is called an *eigenvector*.

   Example: The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues i and $-i$.

   We have $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix} = i\begin{bmatrix} 1 \\ -i \end{bmatrix}$.