**Instructions:** Complete the following exercises.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Monday, March 9**.

1. Draw the crystal graph of $\mathcal{B} = \mathbb{B}_3 \otimes \mathbb{B}_3 \otimes \mathbb{B}_3$.

   Determine the highest weight elements and full subcrystals.

   Express the character $\mathrm{ch}(\mathcal{B})$ as a linear combination of Schur functions.

   You can do this by hand, but you are also welcome (even encouraged) to investigate how to carry out this exercise using the computer algebra system Sage; see:

   `http://doc.sagemath.org/html/en/thematic_tutorials/lie/crystals.html`
   `https://cocalc.com/`

2. Suppose $v = v_1 v_2 \cdots v_m$ and $w = w_1 w_2 \cdots w_m$ are words of the same length.

   We say that $v$ and $w$ are connected by a *Knuth move* if $w$ is obtained from $v$ by applying one of the following transformations to three consecutive letters, assuming $a < b < c$:

   $$cab \leftrightarrow acb, \qquad bac \leftrightarrow bca, \qquad aba \leftrightarrow baa, \qquad bba \leftrightarrow bab$$

   This happens, for example, if $v = 433574$ and $w = 343574$ or $w = 433547$.

   *Knuth equivalence* is the equivalence relation on words that has $v \overset{\mathsf{K}}{\sim} w$ if and only if $v$ and $w$ are connected by a sequence of Knuth moves. For example, $43534 \overset{\mathsf{K}}{\sim} 43354 \overset{\mathsf{K}}{\sim} 34354 \overset{\mathsf{K}}{\sim} 34534$.

   Prove that if $w$ is any word then $w \overset{\mathsf{K}}{\sim} \mathfrak{row}(P_{\mathsf{RSK}}(w))$.

3. The *column reading word* of a tableau $T$ is the word $\mathfrak{col}(T)$ formed by reading the numbers up each column, going left to right. For example,

   $$\mathfrak{col}\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 4 \\ \cline{1-2} 6 & 6 \\ \cline{1-2}\end{array}\right) = 6316422 \qquad \text{while} \qquad \mathfrak{row}\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 4 \\ \cline{1-2} 6 & 6 \\ \cline{1-2}\end{array}\right) = 6634122.$$

   Prove that if $T$ is any semistandard tableau then $\mathfrak{row}(T) \overset{\mathsf{K}}{\sim} \mathfrak{col}(T)$.

4. Prove that if $v$ and $w$ are words with $v \overset{\mathsf{K}}{\sim} w$ then $P_{\mathsf{RSK}}(v) = P_{\mathsf{RSK}}(w)$.

   *Hint: assume $v$, $w$ only differ in their last three letters. Think about how to compute $P_{\mathsf{RSK}}$ by inserting last three letters into first row of $P_{\mathsf{RSK}}(v_1 \cdots v_{m-3}) = P_{\mathsf{RSK}}(w_1 \cdots w_{m-3})$, then use induction.*

5. Suppose $w = w_1 w_2 \cdots w_m$ is a word with each letter $w_i \in \{1, 2, \ldots, n\}$. Fix $i \in [n-1]$.

   Show that if $f_i(w) = 0$ then $f_i(P_{\mathsf{RSK}}(w) = 0$ and otherwise $f_i(P_{\mathsf{RSK}}(w)) = P_{\mathsf{RSK}}(f_i(w))$.

6. Suppose $w = w_1 w_2 \cdots w_m$ is a word with each letter $w_i \in \{1, 2, \ldots, n\}$. Fix $i \in [n-1]$.

   Show that if $f_i(w) \neq 0$ then $Q_{\mathsf{RSK}}(w) = Q_{\mathsf{RSK}}(f_i(w))$.