Instructions: Complete the following exercises.

To receive the highest possible score you should prove any nontrivial statements in your solutions.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Wednesday, March 18**.

1. Show that if $\mathcal{B}$ and $\mathcal{C}$ are crystals of an arbitrary Cartan type, then $(\mathcal{B} \otimes \mathcal{C})^\vee \cong \mathcal{C}^\vee \otimes \mathcal{B}^\vee$.

2. Let $\mathbb{B}_n$ be the standard crystal for Cartan type $\mathrm{GL}(n)$. Write $\mathbb{B}_n^\vee$ for its dual.
   (a) Find a partition λ with at most $n-1$ parts and some $\eta \in \mathbb{Z}\text{-span}\{\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n\}$ such that
   $$\mathbb{B}_n^\vee \cong \mathrm{SSYT}_n(\lambda) \otimes \mathcal{T}_\eta.$$
   (b) Show that $\mathbb{B}_n^\vee$ is isomorphic to a twist of a full subcrystal of $\mathbb{B}_n^{\otimes(n-1)}$.
   (c) Use these observations to show that if $\mathcal{B}$ is a Stembridge crystal then so is $\mathcal{B}^\vee$.

3. Let $\mathrm{GL}(3)$ be the group of 3×3 invertible complex matrices. Let $V = \mathbb{C}^3$.

   The group $\mathrm{GL}(3)$ acts on V by left multiplication, and V is irreducible as a $\mathrm{GL}(3)$-module.

   The tensor product $V \otimes V$ is a $\mathrm{GL}(3)$-module for the action $g : v_1 \otimes v_2 \mapsto gv_1 \otimes gv_2$.

   Decompose $V \otimes V$ as a direct sum of irreducible submodules.

   Explain how this decomposition relates to the decomposition of $\mathbb{B}_3 \otimes \mathbb{B}_3$ into full subcrystals.

4. Let (Φ, Λ) be a Cartan type in an inner product space V, with positive roots Φ^+, simple roots $\{\alpha_i : i \in I\}$, and fundamental weights $\{\varpi_i : i \in I\}$. Define $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$.
   (a) Show that $\langle \rho - \sum_{i\in I} \varpi_i, \alpha_j^\vee \rangle = 0$ for all $j \in I$.
   (b) Show that if (Φ, Λ) is semisimple (meaing that $V = \mathbb{R}\Phi$), then $\rho = \sum_{i\in I} \varpi_i$.

5. Let $\mathbb{B}_n$ be the standard crystal for Cartan type $\mathrm{GL}(n)$. Write $\mathbb{B}_n^\vee$ for its dual.

   Compute the full subcrystals of $\mathbb{B}_n \otimes \mathbb{B}_n^\vee$. (That is, identify the twists of crystals of semistandard tableaux that are isomorphic to each full subcrystal.)

6. Let $\mathbb{B}$ be the standard crystal of type C_n. Show that $\mathbb{B}\otimes\mathbb{B}$ has exactly three highest weight elements, and therefore three full subcrstals. What are the sizes of these subcrystals?