Instructions: Complete the following exercises.

To receive the highest possible score you should prove any nontrivial statements in your solutions.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Monday, March 30**.

1. Let (Φ, Λ) be a Cartan type with simple roots $\{\alpha_i : i \in I\}$. Write V for the ambient vector space.

   Recall that the corresponding Weyl group is $W = \langle s_i : i \in I\rangle$ where $s_i \in \mathrm{GL}(V)$ is the reflection
   $$v \mapsto v - \langle v, \alpha_i^\vee\rangle \alpha_i \qquad \text{where } \alpha_i^\vee = \tfrac{2\alpha_i}{\langle \alpha_i, \alpha_i\rangle}.$$
   This group is finite and contains a unique *longest element* $w_0 = w_0^{-1}$ such that $w_0(\Phi^+) = \Phi^-$.

   (a) Prove that there is a bijection $\tau : I \to I$ such that $w_0(\alpha_i) = -\alpha_{\tau(i)}$ for all $i \in I$.

   (b) Show that if Φ has type A_{n-1} so that $I = \{1, 2, \dots, n-1\}$ then $\tau(i) = n - i$.

   (c) Determine the permutation τ for Cartan types B_n, C_n, and D_n.

   Hint: in type D_n, the answer may depend on the value of n (mod 2).

2. Again let (Φ, Λ) be a Cartan type with simple roots $\{\alpha_i : i \in I\}$.

   Suppose $\mathcal{C}$ is a connected normal crystal of type (Φ, Λ). Define w_0 and τ as in Exercise 1.

   We say that $\mathcal{C}$ has a *crystal involution* if there is a map $S : \mathcal{C} \to \mathcal{C}$ with $\mathbf{wt}(S(x)) = w_0(\mathbf{wt}(x))$ and
   $$e_i(S(x)) = f_{\tau(i)}(x), \quad f_i(S(x)) = e_{\tau(i)}(x), \quad \varepsilon_i(S(x)) = \varphi_{\tau(i)}(x), \quad \varphi_i(S(x)) = \varepsilon_{\tau(i)}(x)$$
   for all $i \in I$ and $x \in \mathcal{B}$. This is sometimes called a *Schützenberger involution* or *Lusztig involution*.

   This exercise shows that such involutions always exist.

   (a) Consider the new crystal structure on $\mathcal{C}$ with weight map $\mathbf{wt}'(x) = w_0(\mathbf{wt}(x))$ and with
   $$e'_i(x) = f_{\tau(i)}(x), \qquad f'_i(x) = e_{\tau(i)}(x), \qquad \varepsilon'_i(x) = \varphi_{\tau(i)}(x), \qquad \varphi'_i(x) = \varepsilon_{\tau(i)}(x)$$
   for all $i \in I$ and $x \in \mathcal{C}$. Show that these operators make $\mathcal{C}$ into a normal crystal.

   If the original highest weight of $\mathcal{C}$ was $\lambda \in \Lambda$, what is the highest weight for this new structure?

   (b) Prove that a connected normal crystal has a unique crystal involution.

   Hint: use the algebraic properties of normal crystals.

   (c) Compute the crystal involutions for the $\mathrm{GL}(3)$ crystals $\mathrm{SSYT}_3(\lambda)$ in the three cases when $\lambda = (1, 0, 0)$, $\lambda = (2, 0, 0)$, and $\lambda = (2, 1, 0)$.

3. Consider an election with n candidates and m voters.

   Let $\alpha_i(t)$ be the total number of votes cast for candidate i after the first t votes have been counted.

   Define $\lambda_i = \alpha_i(m)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$. This vector records the final outcome of the election.

   Assume the candidates are numbered such that λ is a partition, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

   The election outcome λ is known ahead of time but the order in which the m votes are counted is chosen uniformly at random.

   Show that the probability that $\alpha_1(t) \geq \alpha_2(t) \geq \cdots \geq \alpha_n(t) \geq 0$ for all $0 \leq t \leq m$ is $f^\lambda / \binom{m}{\lambda}$ where f^λ is the number of standard tableaux of shape λ and $\binom{m}{\lambda} = \frac{m!}{\lambda_1!\lambda_2!\cdots\lambda_n!}$.

4. A skew shape λ/μ is a *vertical strip* if $\mathsf{D}_{\lambda/\mu}$ has no two boxes in the same row.

   Show that if μ is a partition of k with at most n parts and $r \leq n$ then

$$\mathrm{SSYT}_n(\mu) \otimes \mathrm{SSYT}_n(1^r) \cong \bigsqcup_{\lambda} \mathrm{SSYT}_n(\lambda)$$

   where the union is over partitions λ of $k + r$ with at most n parts such that λ/μ is a vertical strip.

   Here $(1^r) = (1, 1, \ldots, 1)$ denotes the partition with r parts of size one.

   Hint: you may want to proceed as in Exercise 8.5 in Bump and Schilling's book.