Instructions: Complete the following exercises.

To receive the highest possible score you should prove any nontrivial statements in your solutions.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Monday, April 20**.

1. Let $\lambda = (3, 2, 1)$ and $\mu = \nu = (2, 1)$.
   (a) Construct all skew tableaux of shape λ/μ with weight ν whose reading words are Yamanouchi.
   (b) Consider the $\mathrm{GL}(2) \times \mathrm{GL}(2)$ crystal obtained by branching $\mathrm{SSYT}_4(\lambda)$.

   Identify the full subcrystals of this crystal that are isomorphic to $\mathrm{SSYT}_2(\mu) \times \mathrm{SSYT}_2(\nu)$.
   (c) Identify the full subcrystals of $\mathrm{SSYT}_3(\mu) \otimes \mathrm{SSYT}_3(\nu)$ isomorphic to $\mathrm{SSYT}_3(\lambda)$.

   Conclude that the Littlewood-Richardson coefficient $c^{(3,2,1)}_{(2,1),(2,1)}$ can be computed in 3 different ways.

2. Suppose $v = v_1 v_2 \cdots v_p$ and $w = w_1 w_2 \cdots w_p$ are words of the same length with positive integer letters $v_i, w_i \in \{1, 2, \dots, n-1\}$. Assume these words are reduced words for (different) permutations in S_n, but the difference $w_i - v_i$ is either 0 or 1 for each $i \in [p]$. Prove that $Q_{\mathsf{EG}}(v) = Q_{\mathsf{EG}}(w)$.

3. Let λ be a partition with at most n nonzero parts.

   Let μ be a partition with at most $n - 1$ nonzero parts.

   We say that λ/μ is a horizontal strip if the Young diagram of μ is contained in the Young diagram of λ and the set difference of these diagrams contains at most one box in each column.

   Show that λ/μ is a horizontal strip if and only if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n \geq 0.$$

4. Suppose λ is a partition and $T \in \mathrm{SSYT}_n(\lambda)$.

   Let $\Omega = (1, 2, 1, 3, 2, 1, 4, 3, 2, 1 \dots, n-1, \dots, 3, 2, 1)$.

   We claimed in class that if $n = 3$, so that $\Omega = (1, 2, 1)$, and the Gelfand-Tsetlin pattern of T is

$$\left\{ \begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & a & & b & \\ & & c & & \end{array} \right\},$$

   then the string pattern of T is

$$\mathrm{string}_\Omega(T) = \left[\begin{array}{cc} \lambda_1 + \lambda_2 - a - b & \lambda_1 - a \\ & a - c \end{array} \right].$$

   Prove this result and generalize to arbitrary n.