Instructions: Complete the following exercises.

To receive the highest possible score you should prove any nontrivial statements in your solutions.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Friday, May 8**.

Consider semistandard tableaux of varying shapes λ with entries in $\{1, 2, \ldots, n\}$. Define an equivalence relation on such tableaux as follows. Let T and T' be tableaux of shapes λ and λ'. Write $T \equiv T'$ if for each i with $1 \leq i \leq n$, where R_i and R_i' are the ith rows, the rows R_i and R_i' only differ by some i's at the beginning of the tableau. For example, the tableaux

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3}\end{array} \qquad \text{and} \qquad \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 & 4 \\ \cline{1-4} 3 \\ \cline{1-1}\end{array}$$

are equivalent. We say that a tableau T is *large* if the number of entries in row i that are equal to i is strictly larger than the total number of entries in row $i+1$. Thus, the second tableau given above is larger. If T is a tableau, then let $[T]$ denote its equivalence class.

1. Prove that if T and T' are equivalent large tableaux, then $e_i(T)$ and $e_i(T')$ are equivalent (or both zero), and that $f_i(T)$ and $f_i(T')$ are equivalent (or both zero). Therefore give a crystal structure to the set of equivalence classes by defining $e_i([T]) = [e_i(T')]$ where T' is any large tableau equivalent to T. Define $\mathbf{wt}([T])$ in such a way that $\mathbf{wt}([T]) = 0$ if T has only i's in row i. Verify that this makes the set of equivalence classes of tableaux into a $\mathrm{GL}(n)$ crystal.
2. Prove that the crystal in the previous exercise is isomorphic to $\mathcal{B}_\infty$ for Cartan type $\mathrm{GL}(n)$.
3. Prove that the character of $\mathcal{B}_\infty$ for Cartan type $\mathrm{GL}(n)$ is $\prod_{\alpha \in \Phi^+}(1 - t^{-\alpha})^{-1}$ where

$$\Phi^+ = \{\mathbf{e}_i - \mathbf{e}_j : 1 \leq i < j \leq n\}$$

is the usual set of positive roots in the type A_{n-1} root system.