1 Logistics

- Books and course outline:

  (a) *Crystal Bases: Representations and Combinatorics* by Bump and Schilling

  (b) *Introduction to Quantum Groups and Crystal Bases* by Hong and Kang

  The lectures will largely follow Bump and Schilling's book. We will try to cover Chapters 1–14, perhaps leaving time for more topics at the end of the term. Hong and Kang's book gives more background from representation theory and should be considered a supplementary text.

  I will be posting lecture notes, but it would also be good to obtain Bump and Schilling's book.

- Useful prererequisites:

  Experience with root systems, from a course on Lie algebras for example.

  Familiarity with Weyl groups and Coxeter systems.

  If these areas are lacking, it should be possible to pick up much of what we need along the way.

- Grades will be based on problem sets which will be assigned every 1-2 weeks.

2 Motivation from symmetric functions

Crystals are combinatorial structures — essentially consisting of certain labeled directed graphs with weight maps satisfying a few axioms — related to representations of Lie groups. They are naturally motivated from representation theory, since crystals have notions of characters, tensor products, and branching rules which correspond in a precise sense to analogous operations on representations.

We begin our exploration of crystals in this lecture with a different set of motivations, starting from some very classical results related to symmetric functions.

A *partition* is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. The numbers λ_i are the *parts* of λ. We define $\ell(\lambda) = k$ and $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$, and for $i > k$ we set $\lambda_i = 0$.

The *Young diagram* of a partition λ is the set of integers pairs

$$\mathsf{D}_\lambda = \{(i,j) \in \{1,2,\dots,\ell(\lambda)\} \times \mathbb{Z} : 1 \leq j \leq \lambda_i\}.$$

A helpful way to visualize this set is to identify the pairs $(i,j) \in \mathsf{D}_\lambda$ with boxes positioned in a matrix:

$$\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \\ \cline{1-1}\end{array}$$ corresponds to $\mathsf{D}_\lambda = \{(1,1),(1,2),(2,1),(3,1)\}$ for $\lambda = (2,1,1)$.

A *tableau* of shape λ is a map $T : \mathsf{D}_\lambda \to \{1,2,3,\dots\}$.

We denote the image of $(i,j) \in \mathsf{D}_\lambda$ under a tableau T by T_{ij}.

To draw a tableau, we fill the boxes in D_λ by the corresponding entries T_{ij}.

A tableau T is *semistandard* if its rows are weakly increasing and its columns are strictly increasing, or equivalently if $T_{ij} \leq T_{i,j+1}$ whenever $(i,j),(i,j+1) \in \mathsf{D}_\lambda$ and $T_{ik} < T_{i+1,j}$ whenever $(i,j),(i+1,j) \in \mathsf{D}_\lambda$.

For example, the following are semistandard tableaux of shape $\lambda = (3,2,1)$:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 2 & 4 \\ \hline 7 & 7 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array}$$

Let $\mathrm{SSYT}_n(\lambda)$ denote the set of semistandard tableaux T of shape λ with all entries $T_{ij} \in \{1, 2, \dots, n\}$.

The *weight* of a tableau T is $\mathbf{wt}(T) = (w_1, w_2, \dots)$ where w_i is the number of boxes of T containing i.

Let $x_1, x_2, x_3, \dots$ be commuting indeterminates.

Given an integer sequence $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$, let $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$.

This is a well-defined (Laurent) monomial if only finitely many α_i's are nonzero.

Definition 2.1. The *Schur polynomial* of a partition λ in n variables is

$$s_\lambda(x_1, x_2, \dots, x_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda)} x^{\mathbf{wt}(T)} \in \mathbb{Z}[x_1, x_2, \dots, x_n].$$

Example 2.2. If $\lambda = (2, 1)$ and $n = 3$ then

$$\mathrm{SSYT}_n(\lambda) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \right\}$$

so we have $s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_2 + x_2 x_3^2 + 2x_1 x_2 x_3$.

Observation 2.3. If λ is a partition with more than n parts then $s_\lambda(x_1, x_2, \dots, x_n) = 0$.

Observation 2.4. The polynomial $s_\lambda(x_1, \dots, x_{n-1})$ is obtained from $s_\lambda(x_1, \dots, x_n)$ by setting $x_n = 0$.

The sum over semistandard tableaux is a common and elementary way of defining Schur polynomials, but it makes some important properties less obvious.

A polynomial $f \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ is *symmetric* if we have

$$f(x_1, x_2, \dots, x_n) = f(x_{i_1}, x_{i_2}, \dots, x_{i_n}) \qquad \text{whenever } \{i_1, i_2, \dots, i_n\} = \{1, 2, \dots, n\}.$$

This holds if and only if for each $i \in [n-1]$, swapping the variables x_i and x_{i+1} leaves f unchanged.

Theorem 2.5. Every Schur polynomial $s_\lambda(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ is symmetric.

Let Sym_n denote the set of symmetric polynomials in $\mathbb{Z}[x_1, x_2, \dots, x_n]$.

This subset is a subring and also a free abelian subgroup.

Theorem 2.6. The Schur polynomials $s_\lambda(x_1, x_2, \dots, x_n)$, as λ ranges over all partitions with at most n parts, form a $\mathbb{Z}$-basis for Sym_n.

Theorem 2.7. For each triple of partitions λ, μ, ν, there are nonnegative integer coefficients $c_{\lambda\mu}^\nu \in \mathbb{N}$ such that $s_\lambda(x_1, x_2, \dots, x_n) s_\mu(x_1, x_2, \dots, x_n) = \sum_\nu c_{\lambda\mu}^\nu s_\nu(x_1, x_2, \dots, x_n)$.

Example 2.8. One can show that the product $s_{(2,1)}(x_1, x_2, x_3) s_{(2)}(x_1, x_2, x_3)$ is

$$s_{(4,1)}(x_1, x_2, x_3) + s_{(3,2)}(x_1, x_2, x_3) + s_{(3,1,1)}(x_1, x_2, x_3) + s_{(2,2,1)}(x_1, x_2, x_3).$$

The numbers $c_{\lambda\mu}^\nu$ are the *Littlewood-Richardson coefficients*.

It follows from the previous theorem that these coefficients exist and are integers. It is much harder to see that they are nonnegative and to give an interpretation for these positive quantities.

Studying crystals will provide a theory that streamlines the proofs of these theorems and similar results.

3 Directed graphs

Let I be an arbitrary set of labels.

An *I-labeled directed graph* or *I-labeled digraph* is a pair $\mathcal{G} = (\mathcal{B}, \mathcal{E})$ where $\mathcal{B}$ is a set of *vertices* and $\mathcal{E}$ is a multiset of *edges*, given formally as ordered triples $(b, c, i) \in \mathcal{B} \times \mathcal{B} \times I$, informally written $b \xrightarrow{i} c$.

Below is an I-labeled digraph with labels $I = \{1, 2\}$ and vertices $\mathcal{B} = \left\{\boxed{A}, \boxed{B}, \boxed{C}, \boxed{D}, \boxed{E}\right\}$:

$$\begin{tikzcd}
\boxed{A} \arrow[d, "2"] & \\
\boxed{B} \arrow[d, "1"] \arrow[dr, "2"] & \\
\boxed{D} \arrow[d, "2"] & \boxed{C} \arrow[dl, "1"'] \\
\boxed{E} \arrow[d, "1"] & \\
\boxed{F} &
\end{tikzcd}$$

Here is a way to encode many labeled digraphs. Fix a vertex set $\mathcal{B}$ and let $0 \notin \mathcal{B}$ be an auxiliary element. Assume for each $i \in I$ we are given operators $e_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ and $f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ such that:

(*) If $b, c \in \mathcal{B}$ then $f_i(b) = c$ if and only if $e_i(c) = b$.

The labeled digraph associated to the data $(\mathcal{B}, \{e_i\}_{i\in I}, \{f_i\}_{i\in I})$ is $\mathcal{G} = (\mathcal{B}, \mathcal{E})$ where

$$\mathcal{E} = \left\{ b \xrightarrow{i} c : b, c \in \mathcal{B} \text{ and } f_i(b) = c \right\}.$$

Observation 3.1. The correspondence $(\mathcal{B}, \{e_i\}_{i\in I}, \{f_i\}_{i\in I}) \mapsto \mathcal{G}$ is injective and its image consists of all I-labeled digraphs with the property that at most one edge labeled by $i \in I$ starts at any given vertex and at most one edge labeled by $i \in I$ ends at any given vertex.

For each $i \in I$ and $b \in \mathcal{B}$, the associated digraph $\mathcal{G}$ contains a unique maximal chain of nonzero vertices

$$\cdots \xrightarrow{i} e_i^3(b) \xrightarrow{i} e_i^2(b) \xrightarrow{i} e_i(b) \xrightarrow{i} b \xrightarrow{i} f_i(b) \xrightarrow{i} f_i^2(b) \xrightarrow{i} f_i^3(b) \xrightarrow{i} \cdots$$

Either end of this chain may be finite or infinite. We call this the *i-string* through b.

By convention, we consider both ends of an i-string to be infinite if the i-string is a cycle.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of nonnegative integers.

For each $i \in I$, the *i-string lengths* of $(\mathcal{B}, \{e_i\}_{i\in I}, \{f_i\}_{i\in I})$ are the maps $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{N} \sqcup \{\infty\}$ defined by

$$\varepsilon_i(b) = \max\{k \geq 0 : e_i^k(b) \neq 0\} \qquad \text{and} \qquad \varphi_i(b) = \max\{k \geq 0 : f_i^k(b) \neq 0\}. \tag{3.1}$$

To remember these definitions, note that "epsilon" corresponds to e and "phi" corresponds to f.

The number $\varphi_i(b)$ counts how many arrows you can follow from b before reaching the end of the corresponding i-string, while $\varepsilon_i(b)$ is the analogous number of arrows you can follow backwards.

Observation 3.2. Let $i \in I$ and $b \in \mathcal{B}$. Assume all i-strings for $(\mathcal{B}, \{e_i\}_{i\in I}, \{f_i\}_{i\in I})$ are finite.

(a) If $e_i(b) \neq 0$ then $\varepsilon_i(e_i(b)) = \varepsilon_i(b) - 1$ and $\varphi_i(e_i(b)) = \varphi_i(b) + 1$.

(b) If $f_i(b) \neq 0$ then $\varepsilon_i(f_i(b)) = \varepsilon_i(b) + 1$ and $\varphi_i(f_i(b)) = \varphi_i(b) - 1$.

4 Crystals

We continue to consider labeled digraphs but now exclusively for $I = [n-1] := \{1, 2, \dots, n-1\}$.

The following is our first attempt at defining a *crystal*. Later we will generalize this definition considerably, but we will get a fair amount of mileage out of today's simplified formulation.

Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{Z}^n$, so $\mathbf{e}_1 = (1, 0, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)$, and so on.

Definition 4.1. A *crystal* is a set $\mathcal{B}$ with maps

$$\mathbf{wt} : \mathcal{B} \to \mathbb{Z}^n \quad \text{and} \quad e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \text{ for } i \in [n-1],$$

where $0 \notin \mathcal{B}$ is an auxiliary element, such that if $i \in [n-1]$ then:

(1) Condition (*) holds for $(\mathcal{B}, \{e_i\}_{i\in[n-1]}, \{f_i\}_{i\in[n-1]})$ and all corresponding i-strings are finite.

Equivalently: if $b, b' \in \mathcal{B}$ then $e_i(b) = b'$ if and only if $f_i(b') = b$.

Additionally, the string lengths ε_i, φ_i defined by (3.1) take only finite values.

(2) If $b \in \mathcal{B}$ and $f_i(b) \neq 0$ then $\mathbf{wt}(f_i(b)) - \mathbf{wt}(b) = \mathbf{e}_{i+1} - \mathbf{e}_i$.

(3) If $b \in \mathcal{B}$ then $\varphi_i(b) - \varepsilon_i(b) = \mathbf{wt}(b)_i - \mathbf{wt}(b)_{i+1}$.

The function $\mathbf{wt}$ is the *weight map* of $\mathcal{B}$. The maps e_i and f_i are the *raising* and *lowering crystal operators*.

Unless otherwise specified, we will always use the symbols e_i, f_i, ε_i, φ_i, $\mathbf{wt}$ for the crystal operators, string lengths, and weight map of a crystal $\mathcal{B}$.

Remark. Later we will define a category of crystals for any root system Φ and weight lattice Λ. The definition here corresponds to taking $\Phi = \Phi_{A_{n-1}}$ to be the type A_{n-1} root system and $\Lambda = \mathbb{Z}^n$.

The *crystal graph* of a crystal $\mathcal{B}$ is the labeled digraph with vertex set $\mathcal{B}$ and edges

$$b \xrightarrow{i} c \qquad \text{for each } b, c \in \mathcal{B} \text{ and } i \in [n-1] \text{ with } f_i(b) = c.$$

This is the labeled digraph we associated to $(\mathcal{B}, \{e_i\}_{i\in[n-1]}, \{f_i\}_{i\in[n-1]})$ in the previous section.

The crystal graph does not uniquely specify a crystal $\mathcal{B}$, as the weight map must also be given.

Example 4.2. There is a *standard crystal* $\mathbb{B}_n$ with crystal graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

and weight function $\mathbf{wt}(\boxed{i}) = \mathbf{e}_i$.

This means that $f_1(\boxed{1}) = \boxed{2}$ and $e_1(\boxed{2}) = \boxed{1}$, for example. Indeed, we have

$$\mathbf{wt}(f_1(\boxed{1})) - \mathbf{wt}(\boxed{1}) = \mathbf{wt}(\boxed{2}) - \mathbf{wt}(\boxed{1}) = \mathbf{e}_2 - \mathbf{e}_1$$

as required, along with

$$\varphi_1(\boxed{2}) - \varepsilon_1(\boxed{2}) = 0 - 1 = (0, 1, 0, 0, \dots, 0)_1 - (0, 1, 0, 0, \dots, 0)_2 = \mathbf{wt}(\boxed{2})_1 - \mathbf{wt}(\boxed{2})_2.$$

Checking the crystal axioms for the other indices $i \in [n-1]$ is straightforward.

Example 4.3. Consider again the labeled digraph

$$\begin{tikzcd}
\boxed{A} \arrow[d, "2"] & \\
\boxed{B} \arrow[d, "1"] \arrow[dr, "2"] & \\
\boxed{C} \arrow[d, "2"] & \boxed{D} \arrow[dl, "1"'] \\
\boxed{E} \arrow[d, "1"] & \\
\boxed{F} &
\end{tikzcd}$$

There is a unique crystal for $n = 3$ that has this as its crystal graph and assigns $\mathbf{wt}(\boxed{A}) = (2, 2, 0)$. This structure must have

$$\mathbf{wt}(\boxed{B}) = (2, 1, 1),\ \ \mathbf{wt}(\boxed{C}) = (1, 2, 1),\ \ \mathbf{wt}(\boxed{D}) = (2, 0, 2),\ \ \mathbf{wt}(\boxed{E}) = (1, 1, 2),\ \ \mathbf{wt}(\boxed{F}) = (0, 2, 2).$$

We want to view crystals as a category, which requires us to define morphisms between crystals. Any weight-preserving map between crystals that preserves string lengths and commutes with the relevant crystal operators should be considered to be a morphism. Our definition should be slightly more flexible, since we will want to allow crystals morphisms $\mathcal{B} \to \mathcal{C}$ that map nonzero elements of $\mathcal{B}$ to $0 \notin \mathcal{C}$.

The definition of a *crystal morphism*, roughly speaking, should therefore be a map $\psi : \mathcal{B} \to \mathcal{C}$ that preserves weights and string lengths and commutes with all crystal operators, except in the cases when $\psi(b) = 0$ and these conditions don't make sense. We won't worry too much about the details for now.

What is easier to define is a *crystal isomorphism*: this is a bijection $\psi : \mathcal{B} \to \mathcal{C}$ between crystals such that

$$\mathbf{wt}(\psi(b)) = \mathbf{wt}(b), \qquad e_i(\psi(b)) = \psi(e_i(b)), \qquad \text{and} \qquad f_i(\psi(b)) = \psi(f_i(b))$$

for all $b \in \mathcal{B}$ and $i \in [n-1]$, where we set $\psi(0) = 0 \notin \mathcal{C}$.

The category of crystals has a tensor product, which we will use to construct new crystals.

Theorem 4.4. Let $\mathcal{B}$ and $\mathcal{C}$ be crystals. The set

$$\mathcal{B} \otimes \mathcal{C} := \{b \otimes c : b \in \mathcal{B},\ c \in \mathcal{C}\}$$

has a unique crystal structure with weight map

$$\mathbf{wt}(b \otimes c) := \mathbf{wt}(b) + \mathbf{wt}(c) \in \mathbb{Z}^n$$

and crystal operators defined by

$$f_i(b \otimes c) := \begin{cases} b \otimes f_i(c) & \text{if } \varepsilon_i(b) < \varphi_i(c) \\ f_i(b) \otimes c & \text{if } \varepsilon_i(b) \geq \varphi_i(c) \end{cases} \qquad \text{and} \qquad e_i(b \otimes c) := \begin{cases} b \otimes e_i(c) & \text{if } \varepsilon_i(b) \leq \varphi_i(c) \\ e_i(b) \otimes c & \text{if } \varepsilon_i(b) > \varphi_i(c) \end{cases}$$

for $i \in [n-1]$, $b \in \mathcal{B}$, and $c \in \mathcal{C}$, where it is understood that $b \otimes 0 = 0 \otimes c = 0$.

Proof. We will skip this mostly straightforward proof. For details, see Bump and Schilling's book. □

Example 4.5. If $n = 3$ then the crystal graph of $\mathbb{B}_n \otimes \mathbb{B}_n$ is

$$\begin{tikzcd}
\boxed{1}\otimes\boxed{1} \arrow[r, "1"] & \boxed{1}\otimes\boxed{2} \arrow[r, "2"] \arrow[d, "1"] & \boxed{1}\otimes\boxed{3} \arrow[d, "1"] \\
\boxed{2}\otimes\boxed{1} \arrow[d, "2"'] & \boxed{2}\otimes\boxed{2} \arrow[r, "2"] & \boxed{2}\otimes\boxed{3} \arrow[d, "2"] \\
\boxed{3}\otimes\boxed{1} \arrow[r, "1"] & \boxed{3}\otimes\boxed{2} & \boxed{3}\otimes\boxed{3}
\end{tikzcd}$$

Here the weight map satisfies $\mathbf{wt}\left(\boxed{i}\otimes\boxed{j}\right) = \mathbf{e}_i + \mathbf{e}_j$.

A *weakly connected component* of a labeled directed graph is a connected component of the simple graph obtained by forgetting all labels and the orientations of all edges. A subset of a crystal $\mathcal{B}$ is a *full subcrystal* if it is a weakly connected component of its crystal graph. Each full subcrystal inherits its own crystal structure. We saw in the previous example that $\mathbb{B}_3 \otimes \mathbb{B}_3$ has 2 full subcrystals with sizes 3 and 6.

Theorem 4.6. The natural maps $\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}) \to (\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}$ are crystal isomorphisms.

Proof. We will also skip this proof. For details, see Bump and Schilling's book. □

Thus, we can dispense with all parentheses in iterated tensor products of crystals.

A crystal $\mathcal{B}$ is *finite* if it has a finite number of vertices.

The *character* of a finite crystal $\mathcal{B}$ is the (Laurent) polynomial $\mathrm{ch}(\mathcal{B}) = \sum_{b\in\mathcal{B}} x^{\mathbf{wt}(b)} \in \mathbb{Z}[x_1^{\pm1}, x_2^{\pm1}, \dots, x_n^{\pm1}]$.

Observation 4.7. If $\mathcal{B}$ and $\mathcal{C}$ are two finite crystals then $\mathrm{ch}(\mathcal{B} \otimes \mathcal{C}) = \mathrm{ch}(\mathcal{B})\mathrm{ch}(\mathcal{C})$.

For example, the character of the standard crystal $\mathbb{B}_n$ is

$$\mathrm{ch}(\mathbb{B}_n) = x_1 + x_2 + \cdots + x_n = s_{(1)}(x_1, x_2 \dots, x_n) \in \mathsf{Sym}_n.$$

The character of $\mathbb{B}_3 \otimes \mathbb{B}_3$ is

$$\mathrm{ch}(\mathbb{B}_3 \otimes \mathbb{B}_3) = (x_1 + x_2 + x_3)(x_1 + x_2 + x_3) \in \mathsf{Sym}_3.$$

Finally, the crystal in Example 4.3 is

$$x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^2x_1 + x_1^2x_3^2 + x_1x_2x_3^2 + x_2^2x_3^2 \in \mathsf{Sym}_3.$$

This function turns out to be the Schur polynomial $s_{(2,2)}(x_1, x_2, x_3)$.

Goals for Lectures 2 and 3:

- Describe the crystal operators for the m-fold tensor product $\mathbb{B}_n^{\otimes m}$.
- Give $\mathrm{SSYT}_n(\lambda)$ a crystal structure via a natural embedding $\mathrm{SSYT}_n(\lambda) \hookrightarrow \mathbb{B}_n^{\otimes|\lambda|}$.
- Show that the character of *any* finite crystal is symmetric.
- Show that any full subcrystal of $\mathbb{B}_n^{\otimes m}$ is isomorphic to $\mathrm{SSYT}_n(\lambda)$ for some λ.
- Finally, deduce the theorems in Section 2.