1 Last time: the category of (type A) crystals

Fix a positive integer n. Let $[n-1] = \{1, 2, \dots, n-1\}$.

Definition 1.1. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{Z}^n$. A *crystal* is a set $\mathcal{B}$ with maps

$$\mathbf{wt} : \mathcal{B} \to \mathbb{Z}^n \quad \text{and} \quad e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \text{ for } i \in [n-1],$$

where $0 \notin \mathcal{B}$ is an auxiliary element, such that if $i \in [n-1]$ then:

(1a) If $b, b' \in \mathcal{B}$ then $e_i(b) = b'$ if and only if $f_i(b') = b$.

(1b) Only finite values are assumed by the string lengths $\varepsilon_i, \varphi_i : \mathcal{B} \to \{0, 1, 2, \dots\}$ defined by

$$\varepsilon_i(b) := \max\{k \geq 0 : e_i^k(b) \neq 0\} \qquad \text{and} \qquad \varphi_i(b) := \max\{k \geq 0 : f_i^k(b) \neq 0\}.$$

(2a) If $b \in \mathcal{B}$ and $f_i(b) \neq 0$ then $\mathbf{wt}(b) - \mathbf{wt}(f_i(b)) = \mathbf{e}_i - \mathbf{e}_{i+1}$.

(2b) If $b \in \mathcal{B}$ then $\mathbf{wt}(b)_i - \mathbf{wt}(b)_{i+1} = \varphi_i(b) - \varepsilon_i(b)$.

The function $\mathbf{wt}$ is the *weight map* of $\mathcal{B}$. The maps e_i and f_i are the *raising* and *lowering crystal operators*.

The *crystal graph* of a crystal $\mathcal{B}$ is the labeled directed graph with vertex set $\mathcal{B}$ and edges

$$b \xrightarrow{i} c \qquad \text{for each } b, c \in \mathcal{B} \text{ and } i \in [n-1] \text{ with } f_i(b) = c.$$

Conditions (1a), (1b) just tell us that the crystal graph has at most one edge labeled by i starting at a given vertex, and has no cycles or infinite paths consisting of edges with the same label.

A *crystal isomorphism* is a bijection $\mathcal{B} \to \mathcal{C}$ between crystals that is weight-preserving and string length-preserving, and that commutes with all crystal operators. Crystals have a tensor product:

Definition 1.2. Let $\mathcal{B}$ and $\mathcal{C}$ be crystals. The set $\mathcal{B} \otimes \mathcal{C} := \{b \otimes c : b \in \mathcal{B},\ c \in \mathcal{C}\}$ has a unique crystal structure with weight map $\mathbf{wt}(b \otimes c) := \mathbf{wt}(b) + \mathbf{wt}(c) \in \mathbb{Z}^n$ and crystal operators defined by

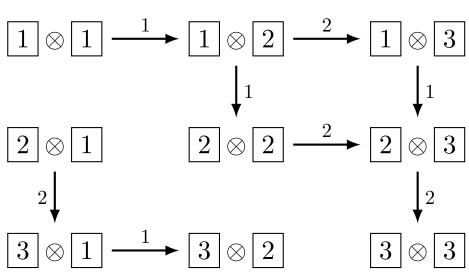
$$f_i(b \otimes c) := \begin{cases} b \otimes f_i(c) & \text{if } \varepsilon_i(b) < \varphi_i(c) \\ f_i(b) \otimes c & \text{if } \varepsilon_i(b) \geq \varphi_i(c) \end{cases} \qquad \text{and} \qquad e_i(b \otimes c) := \begin{cases} b \otimes e_i(c) & \text{if } \varepsilon_i(b) \leq \varphi_i(c) \\ e_i(b) \otimes c & \text{if } \varepsilon_i(b) > \varphi_i(c). \end{cases}$$

Here, $b \otimes 0 := 0 =: 0 \otimes c$. The natural maps $\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}) \to (\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}$ are crystal isomorphisms.

Example 1.3. There is a *standard crystal* $\mathbb{B}_n$ with crystal graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

and weight function $\mathbf{wt}(\boxed{i}) = \mathbf{e}_i$. The crystal graph of the tensor product $\mathbb{B}_3 \otimes \mathbb{B}_3$ is



The weakly connected components of the crystal graph are called *full subcrystals*.

The *character* of a finite crystal $\mathcal{B}$ is $\mathrm{ch}(\mathcal{B}) = \sum_{b \in \mathcal{B}} x^{\mathbf{wt}(b)} \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$.

If $\mathcal{B}$ and $\mathcal{C}$ are two finite crystals then $\mathrm{ch}(\mathcal{B} \otimes \mathcal{C}) = \mathrm{ch}(\mathcal{B})\mathrm{ch}(\mathcal{C})$.

2 Symmetry

Conditions (2a) and (2b) in the definition of a crystal subtly impose a lot of structure.

Let $s_i = (i, i+1)$ be the permutation of $[n] := \{1, 2, \dots, n\}$ that interchanges i and $i+1$.

The elements $s_1, s_2, \dots, s_{n-1}$ generate the group S_n of all permutations of $[n]$.

There is a unique (left) group action of S_n on $\mathbb{Z}^n$ with

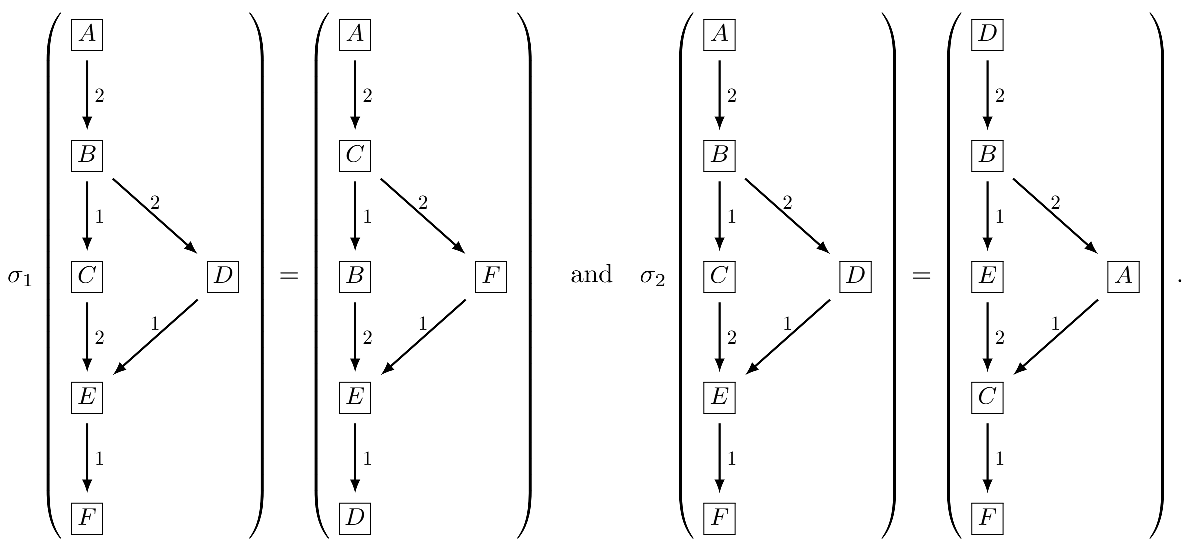
$$s_i(\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_n) = (\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_n).$$

Let $\mathcal{B}$ be a crystal. For each $i \in [n-1]$ we define a map $\sigma_i : \mathcal{B} \to \mathcal{B}$ as follows:

Definition 2.1. Given $x \in \mathcal{B}$, let $k = \mathbf{wt}(x)_i - \mathbf{wt}(x)_{i+1} = \varphi_i(x) - \varepsilon_i(x)$ and define

$$\sigma_i(x) = \begin{cases} f_i^k(x) & \text{if } k > 0, \\ x & \text{if } k = 0, \\ e_i^{-k}(x) & \text{if } k < 0. \end{cases}$$

For example,



Proposition 2.2. The map σ_i is a self-inverse bijection $\mathcal{B} \to \mathcal{B}$ satisfying $\mathbf{wt}(\sigma_i(x)) = s_i(\mathbf{wt}(x))$.

The map σ_i has the effect of reversing each i-string in $\mathcal{B}$.

For example, for an i-string of the form

$$\begin{array}{cccccccccc} & b_1 & \xrightarrow{i} & b_2 & \xrightarrow{i} & b_3 & \xrightarrow{i} & b_4 & \xrightarrow{i} & b_5 \\ k = & 4 & & 2 & & 0 & & -2 & & -4 \end{array}$$

we have $\sigma_i(b_j) = b_{6-j}$.

Proof. Fix $x \in \mathcal{B}$ and define $k = \mathbf{wt}(x)_i - \mathbf{wt}(x)_{i+1} = \varphi_i(x) - \varepsilon_i(x)$.

We can apply f_i to x exactly k times before reaching 0 if $k > 0$, and e_i to x exactly $-k$ times if $k < 0$.

Thus σ_i defines a map $\mathcal{B} \to \mathcal{B}$ and in either case $\mathbf{wt}(\sigma_i(x)) = \mathbf{wt}(x) - k\mathbf{e}_i + k\mathbf{e}_{i+1} = s_i(\mathbf{wt}(x))$.

Within the i-string through x, there is a unique element of weight $s_i(\mathbf{wt}(x))$, so σ_i reverses the i-string end to end and is a self-inverse bijection. ∎

There is also a unique action of the symmetric group S_n on $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ with $s_i x_j^{\pm 1} = x_{s_i(j)}^{\pm 1}$.

The symmetric polynomials in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ are the elements fixed by this action.

Proposition 2.3. The character of a finite crystal $\mathcal{B}$ is a symmetric polynomial in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$.

Proof. For each $i \in [n-1]$, since σ_i is a bijection, we have

$$\mathrm{ch}(\mathcal{B}) := \sum_{b\in\mathcal{B}} x^{\mathbf{wt}(b)} = \sum_{b\in\mathcal{B}} x^{\mathbf{wt}(\sigma_i(b))} = \sum_{b\in\mathcal{B}} x^{s_i(\mathbf{wt}(b))} = s_i(\mathrm{ch}(\mathcal{B})).$$

□

3 Crystals of words

We can form the standard crystal $\mathbb{B}_n$, take tensor products, and restrict to full subcrystals.

Perhaps surprisingly, it will be possible to describe everything that these operations generate.

As a first step, we need to understand the tensor products $\mathbb{B}_n^{\otimes m}$ for all positive integers m, n.

We use the term *word* to mean a finite sequence of positive integers $w = w_1 w_2 \cdots w_m$.

Identify the tensors $w_1 \otimes w_2 \otimes \cdots \otimes w_m$ (which are the elements of $\mathbb{B}_n^{\otimes m}$) with words $w = w_1 w_2 \cdots w_m$.

Clearly $\mathbf{wt}(w)$ is the n-tuple whose ith entry is the number of occurrences of i in w.

Fix $i \in [n-1]$. Replace each i in w by a right parenthesis and each $i+1$ in w by a left parenthesis:

$$w = 1223343212 \text{ and } i = 2 \quad \rightsquigarrow \quad \left[\begin{array}{cccccccccc} 1 &) &) & (& (& 4 & (&) & 1 &) \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}\right].$$

We have the following *signature rules* for the crystal operators f_i and e_i in $\mathbb{B}_n^{\otimes m}$.

Proposition 3.1. To apply the crystal operator f_i of $\mathbb{B}_n^{\otimes m}$ to w, consider the parenthesized word just described. If each right parenthesis ")" belongs to a balanced pair, then $f_i(w) = 0$. Otherwise, form $f_i(w)$ from w by changing the letter i corresponding to the **last** unbalanced right parenthesis to $i+1$.

Proposition 3.2. To apply the crystal operator e_i of $\mathbb{B}_n^{\otimes m}$ to w, again consider the parenthesized word. If each left parenthesis "(" belongs to a balanced pair, then $e_i(w) = 0$. Otherwise, form $e_i(w)$ by changing the $i+1$ in w corresponding to the **first** unbalanced left parenthesis to i.

Here is our running example:

$$w = 1223343212 \text{ and } i = 2 \quad \rightsquigarrow \quad \left[\begin{array}{cccccccccc} 1 &) &) & (& (& 4 & (&) & 1 &) \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}\right] \quad \rightsquigarrow \quad \begin{cases} f_2(w) = 12\underline{3}3343212 \\ e_2(w) = 122\underline{2}343212. \end{cases}$$

Let's examine a few other cases in detail before proving the propositions.

Example 3.3. If $w = w_1 = j \in \mathbb{B}_n^{\otimes 1}$ is a single letter then the parenthesized word is

- $\left[\begin{array}{c}) \\ 1 \end{array}\right]$ if $i = j$, in which case $f_i(w) = j+1$ and $e_i(w) = 0$.
- $\left[\begin{array}{c} (\\ 1 \end{array}\right]$ if $i+1 = j$, in which case $f_i(w) = 0$ and $e_i(w) = j-1$.

- $\left[\begin{smallmatrix} j \\ 1 \end{smallmatrix}\right]$ if $i \notin \{j-1, j\}$, in which case $f_i(w) = 0$ and $e_i(w) = 0$.

Thus, in this case the rules for f_i and e_i agree with the crystal graph for $\mathbb{B}_n = \mathbb{B}_n^{\otimes 1}$:

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

Example 3.4. If $w = jj\cdots j \in \mathbb{B}_n^{\otimes m}$ is one repeated letter, then the parenthesized word is

- $\left[\begin{smallmatrix}) &) & \dots &) \\ 1 & 2 & \dots & m \end{smallmatrix}\right]$ if $i = j$, in which case w is at the start of an i-string with m arrows:

$$0 \xleftarrow{e_i} w = i\cdots ii \xrightarrow{f_i} i\cdots i(i+1) \xrightarrow{f_i} i\cdots(i+1)(i+1) \xrightarrow{f_i} \cdots \xrightarrow{f_i} (i+1)\cdots(i+1)(i+1).$$

- $\left[\begin{smallmatrix} (& (& \dots & (\\ 1 & 2 & \dots & m \end{smallmatrix}\right]$ if $i + 1 = j$, in which case w is the end of an i-string with m arrows:

$$0 \xleftarrow{f_i} w = (i+1)(i+1)\cdots(i+1) \xrightarrow{e_i} i(i+1)\cdots(i+1) \xrightarrow{e_i} ii\cdots(i+1) \xrightarrow{e_i} \cdots \xrightarrow{e_i} ii\cdots i.$$

- $\left[\begin{smallmatrix} j & j & \dots & j \\ 1 & 2 & \dots & m \end{smallmatrix}\right]$ if $i \notin \{j-1, j\}$, in which case $f_i(w) = 0$ and $e_i(w) = 0$.

Proof. We prove the propositions by induction on the number of tensor factors m.

For $m = 1$, we saw in the examples that the signature rule agrees with the standard crystal graph.

Now fix $i \in [n-1]$ and consider a word $w = w_1w_2\cdots w_mw_{m+1} \in \mathbb{B}_n^{\otimes(m+1)}$.

If $w_{m+1} \neq i$, then $\varepsilon_i(w_1w_2\cdots w_m) \geq 0 = \varphi_i(w_{m+1})$, so we expect to have

$$f_i(w) = f_i(w_1w_2\cdots w_m)w_{m+1}.$$

This holds by induction since if $w_{m+1} \neq i$ then adding w_{m+1} has no effect on the unbalanced right parentheses associated to $w_1w_2\cdots w_m$.

Assume $w_{m+1} = i$. Then $\varphi_i(w_{m+1}) = 1$, so (by induction) the only way we can have $\varepsilon_i(w_1w_2\cdots w_m) < \varphi_i(w_{m+1})$ is if there are no unbalanced left parentheses in the word associated to $w_1w_2\cdots w_m$. If this happens then w_{m+1} will contribute the last unbalanced right parenthesis in w and we will have

$$f_i(w) = w_1w_2\cdots w_mf_i(w_{m+1}),$$

as desired. Otherwise, there will be at least one unbalanced left parenthesis in the word associated to $w_1w_2\cdots w_m$, so the right parenthesis contributed by w_{m+1} will be part of a balanced pair and

$$f_i(w) = f_i(w_1w_2\cdots w_m)w_{m+1}$$

by induction, as needed.

The argument that the tensor product formula for e_i coincides with the signature rule is similar. □

4 Crystals of tableaux

Recall from last time: a *partition* is a sequence of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$.

The *Young diagram* of a partition λ is the set $\mathsf{D}_\lambda = \{(i,j) \in \{1,2,\ldots,\ell(\lambda)\} \times \mathbb{Z} : 1 \leq j \leq \lambda_i\}$

A *tableau* of shape λ is a map $T : \mathsf{D}_\lambda \to \{1,2,3,\ldots\}$, written $(i,j) \mapsto T_{ij}$.

To draw a tableau, we fill the boxes in D_λ, oriented using matrix coordinates, by the entries T_{ij}.

A tableau T is *semistandard* if its rows are weakly increasing and its columns are strictly increasing.

Let $\mathrm{SSYT}_n(\lambda)$ denote the set of semistandard tableaux T of shape λ with all entries $T_{ij} \in \{1, 2, \dots, n\}$.

"SSYT" stands for "semistandard Young tableaux."

For $n = 3$ and $\lambda = (2, 1)$, we have

$$\mathrm{SSYT}_n(\lambda) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \right\}.$$

The set $\mathrm{SSYT}_n(\lambda)$ is empty if λ has more than n parts.

Goal: give $\mathrm{SSYT}_n(\lambda)$ a crystal structure for any partition λ with $\ell(\lambda) \leq n$.

The *row reading word* of a tableau T is the word $\mathfrak{row}(T)$ formed by concatenating the rows of T in reverse order, i.e., starting with the bottom row. For example, if

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 & 4 & 5 \\ \cline{1-3} 7 & 7 \\ \cline{1-2}\end{array} \qquad \text{then} \qquad \mathfrak{row}(T) = 7734523412234.$$

Proposition 4.1. The row reading word $\mathfrak{row}$ is an injective map from $\mathrm{SSYT}_n(\lambda)$ to the set of words of length $|\lambda|$ with letters in $\{1, 2, \dots, n\}$, which we identify with $\mathbb{B}_n^{\otimes|\lambda|}$.

Proof. You can recover the rows of T by dividing $\mathfrak{row}(T)$ into maximal weakly increasing subwords.

For example, $7734523412234 \rightsquigarrow 77 \mid 345 \mid 234 \mid 12234$. □

Proposition 4.2. Fix $T \in \mathrm{SSYT}_n(\lambda)$ and $i \in [n-1]$.

(a) If $f_i(\mathfrak{row}(T)) \neq 0$ then there is a unique $f_i(T) \in \mathrm{SSYT}_n(\lambda)$ with $\mathfrak{row}(f_i(T)) = f_i(\mathfrak{row}(T)) \in \mathbb{B}_n^{\otimes|\lambda|}$.

(b) If $e_i(\mathfrak{row}(T)) \neq 0$ then there is a unique $e_i(T) \in \mathrm{SSYT}_n(\lambda)$ with $\mathfrak{row}(e_i(T)) = e_i(\mathfrak{row}(T)) \in \mathbb{B}_n^{\otimes|\lambda|}$.

Proof. Changing the letter corresponding to the last unbalanced right parenthesis in $\mathfrak{row}(T)$ from i to $i+1$ gives the row reading word of a tableau formed from T by changing the last i in some row to $i+1$:

$$\begin{array}{|c|c|c|}\hline \cdots & i & j \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline \cdots & i+1 & j \\ \hline\end{array}.$$

The only way this could fail to be semistandard is if the changed letter is directly above another $i+1$:

$$\begin{array}{|c|c|c|}\hline \cdots & i & j \\ \hline \cdots & i+1 & \cdots \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline \cdots & i+1 & j \\ \hline \cdots & i+1 & \cdots \\ \hline\end{array}.$$

But column strictness implies that this can only happen if these two rows in our tableau have the form

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & h & i & \cdots & i & j & \cdots \\ \hline \cdots & x & i+1 & \cdots & i+1 & y & \cdots \\ \hline\end{array}$$

where $h < i < j$ and $x \leq i+1 < y$. But then all i's in the first row are balanced right parentheses. This proves part (a). The argument for part (b) is similar. □

Thus, there are unique maps $e_i, f_i : \mathrm{SSYT}_n(\lambda) \to \mathrm{SSYT}_n(\lambda) \sqcup \{0\}$ for $i \in [n-1]$ such that

$$\mathfrak{row}(e_i(T)) = e_i(\mathfrak{row}(T)) \qquad \text{and} \qquad \mathfrak{row}(f_i(T)) = f_i(\mathfrak{row}(T)),$$

where we interpret $\mathfrak{row}(0) = 0$. Last time we defined $\mathbf{wt}(T) \in \mathbb{Z}^n$ for $T \in \mathrm{SSYT}_n(\lambda)$, and clearly

$$\mathbf{wt}(T) = \mathbf{wt}(\mathfrak{row}(T)).$$

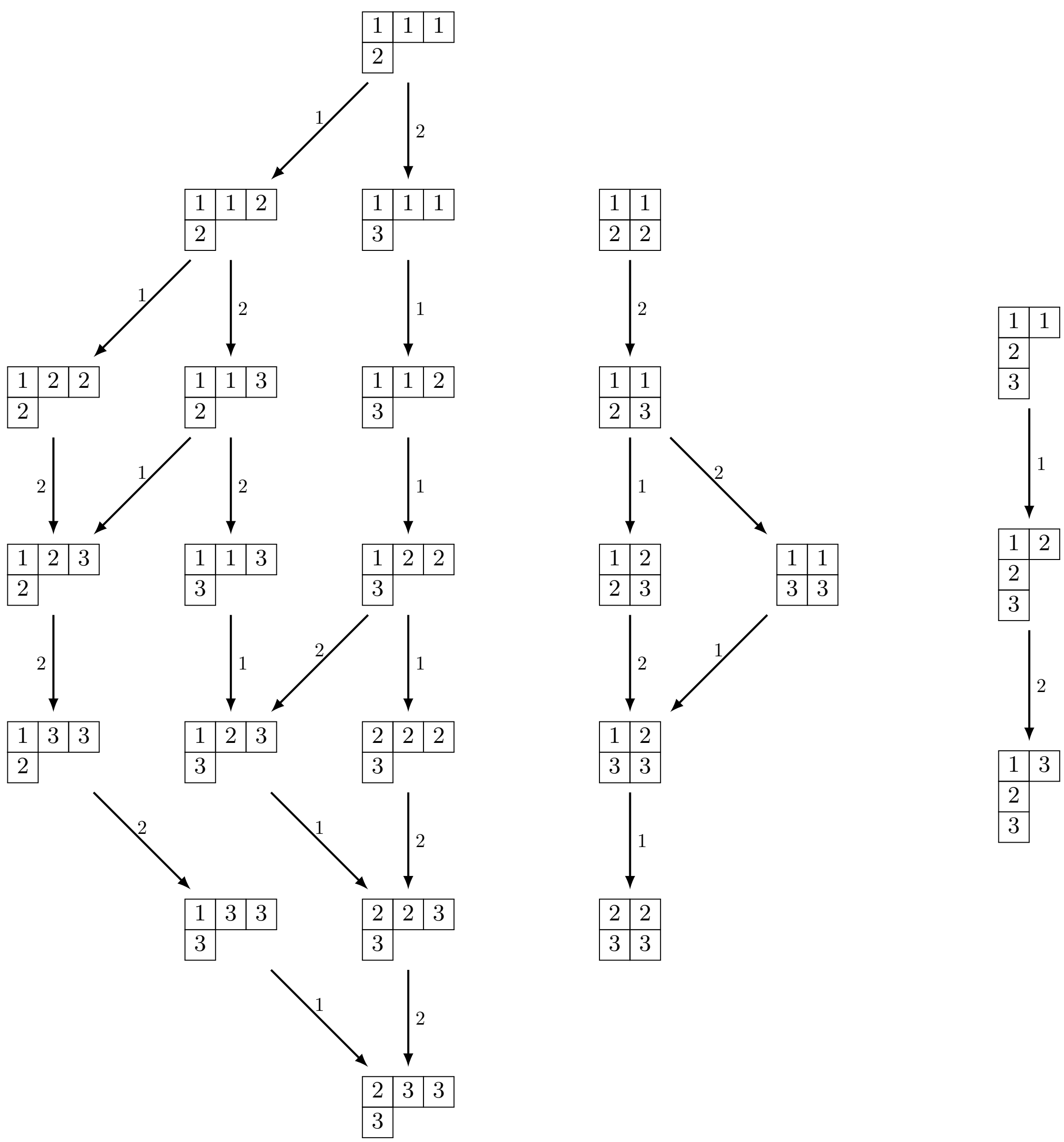
Theorem 4.3. Suppose λ is a partition with at most n parts.

With respect to $\mathbf{wt}$, e_i, f_i, the set $\mathrm{SSYT}_n(\lambda)$ is a crystal whose crystal graph is weakly connected.

The map $\mathfrak{row}$ is an injective crystal morphism that identifies $\mathrm{SSYT}_n(\lambda)$ with a full subcrystal of $\mathbb{B}_n^{\otimes|\lambda|}$.

Before justifying this result, let us examine some of the crystals $\mathrm{SSYT}_n(\lambda)$.

Below are the crystal graphs of $\mathrm{SSYT}_3(\lambda)$ for $\lambda \in \{(3,1),(2,2),(2,1,1)\}$.



To prove the theorem, it only remains to show that the crystal graph of $\mathrm{SSYT}_n(\lambda)$ is weakly connected.

To this end, we will use the following terminology. A *highest weight* of a crystal $\mathcal{B}$ is an element $b \in \mathcal{B}$ with $e_i(b) = 0$ for all $i \in [n-1]$. Such an element is not the target of any edge in the crystal graph.

A highest weight element in $\mathrm{SSYT}_n(\lambda)$ is the tableau T_λ with all entries in row i equal to i. For example,

$$T_\lambda = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 & 3 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} \end{array} \qquad \text{for } \lambda = (5,3,3,2).$$

This is because, for each $i \in [n-1]$, in the word associated to $\mathfrak{row}(T)$ all left parentheses are balanced.

To prove that $\mathrm{SSYT}_n(\lambda)$ is weakly connected, it suffices to prove this lemma:

Lemma 4.4. The tableau T_λ is the unique highest weight in $\mathrm{SSYT}_n(\lambda)$.

Proof. Let $T \in \mathrm{SSYT}_n(\lambda) - \{T_\lambda\}$. Then T has a row containing an entry greater than the row index.

Suppose the largest entry in the first such row is $i+1$.

All letters after $i+1$ in $\mathfrak{row}(T)$ will be less than i, so $e_i(T) \neq 0$ and T is not a highest weight. $\square$

Observe that the $\mathbf{wt}(T_\lambda) = \lambda$.

Recall that the *Schur polynomial* of λ is $s_\lambda(x_1, x_2, \dots, x_n) := \sum_{T \in \mathrm{SSYT}_n(\lambda)} x^{\mathbf{wt}(T)}$.

As corollary of the results today, we recover the theorem from last time:

Corollary 4.5. The Schur polynomial $s_\lambda(x_1, x_2, \dots, x_n)$ is symmetric, since it is the character of the connected crystal of semistandard tableaux $\mathrm{SSYT}_n(\lambda)$.

Next time: every full subcrystal of $\mathbb{B}_n^{\otimes m}$ is isomorphic to the crystal $\mathrm{SSYT}_n(\lambda)$ for some partition λ.