1 Review from last time

1.1 Characters of crystals are symmetric

Fix a positive integer n and set $[n-1] = \{1, 2, \dots, n-1\}$. Our working definition of a crystal:

Definition 1.1. A *crystal* is a set $\mathcal{B}$ with maps $\mathbf{wt} : \mathcal{B} \to \mathbb{Z}^n$ and $e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ for $i \in [n-1]$, where $0 \notin \mathcal{B}$ is an auxiliary element, satisfying some fairly simple axioms, including in particular:

- If $b, b' \in \mathcal{B}$ then $e_i(b) = b'$ if and only if $f_i(b') = b$, in which case $\mathbf{wt}(b') - \mathbf{wt}(b) = \mathbf{e}_i - \mathbf{e}_{i+1}$.
- If $b \in \mathcal{B}$ then $\mathbf{wt}(b)_i - \mathbf{wt}(b)_{i+1} = \varphi_i(b) - \varepsilon_i(b)$, where

$$\varepsilon_i(b) := \max\{k \geq 0 : e_i^k(b) \neq 0\} \qquad \text{and} \qquad \varphi_i(b) := \max\{k \geq 0 : f_i^k(b) \neq 0\}.$$

Each crystal determines a *crystal graph* consisting of the edges $b \xrightarrow{i} f_i(b) \neq 0$ for $b \in \mathcal{B}$.

The weakly connected components of this graph are called *full subcrystals*.

Example 1.2. There is a *standard crystal* $\mathbb{B}_n$ with weight function $\mathbf{wt}(\boxed{i}) = \mathbf{e}_i$ and crystal graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

The *character* of a finite crystal $\mathcal{B}$ is $\mathrm{ch}(\mathcal{B}) = \sum_{b \in \mathcal{B}} x^{\mathbf{wt}(b)} \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$.

Crystals have a natural tensor product $\otimes$.

If $\mathcal{B}$ and $\mathcal{C}$ are two finite crystals then $\mathrm{ch}(\mathcal{B} \otimes \mathcal{C}) = \mathrm{ch}(\mathcal{B})\mathrm{ch}(\mathcal{C})$.

Proposition 1.3. The character of a finite crystal $\mathcal{B}$ is a **symmetric** Laurent polynomial.

1.2 Signature rule

Last time we described the crystal operators for $\mathbb{B}_n^{\otimes m}$.

Identify tensors $w_1 \otimes w_2 \otimes \cdots \otimes w_m$ in $\mathbb{B}_n^{\otimes m}$ with words $w = w_1 w_2 \cdots w_m$.

The value of $\mathbf{wt}(w)$ is the n-tuple whose ith entry is the number of occurrences of i in w.

Fix $i \in [n-1]$. Replace each i in w by a right parenthesis and each $i+1$ in w by a left parenthesis:

$$w = 1223343212 \text{ and } i = 2 \quad \rightsquigarrow \quad \left[\begin{array}{cccccccccc} 1 &) &) & (& (& 4 & (&) & 1 &) \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}\right].$$

We have the following *signature rules* for the crystal operators f_i and e_i in $\mathbb{B}_n^{\otimes m}$.

- To apply the crystal operator f_i of $\mathbb{B}_n^{\otimes m}$ to w, consider the parenthesized word just described. If each right parenthesis ")" belongs to a balanced pair, then $f_i(w) = 0$. Otherwise, form $f_i(w)$ from w by changing the letter i corresponding to the **last** unbalanced right parenthesis to $i+1$.
- To apply the crystal operator e_i of $\mathbb{B}_n^{\otimes m}$ to w, again consider the parenthesized word. If each left parenthesis "(" belongs to a balanced pair, then $e_i(w) = 0$. Otherwise, form $e_i(w)$ by changing the $i+1$ in w corresponding to the **first** unbalanced left parenthesis to i.

These rules mean that $f_3(33333) = 33334$ and $e_3(44444) = 34444$, for example.

1.3 Crystals of semistandard tableaux

The *Young diagram* of a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ is the set $\mathsf{D}_\lambda = \{(i,j) : 1 \leq j \leq \lambda_i\}$

A *tableau* of shape λ is a map $T : \mathsf{D}_\lambda \to \{1, 2, 3, \dots\}$, written $(i,j) \mapsto T_{ij}$.

A tableau T is *semistandard* if its rows are weakly increasing and its columns are strictly increasing.

Let $\mathrm{SSYT}_n(\lambda)$ denote the set of semistandard tableaux T of shape λ with all entries $T_{ij} \in \{1, 2, \dots, n\}$.

The *row reading word* of T is the word $\mathfrak{row}(T)$ formed by concatenating the rows of T in reverse order:

$$\mathfrak{row}\left(\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}\right) = 322.$$

Last time: we can always recover $T \in \mathrm{SSYT}_n(\lambda)$ from the word $\mathfrak{row}(T)$.

Fix $T \in \mathrm{SSYT}_n(\lambda)$ and view $\mathfrak{row}(T) \in \mathbb{B}_n^{\otimes|\lambda|}$. Last time: if we apply e_i or f_i to $\mathfrak{row}(T)$ using the signature rule, then the result is either zero or the row reading word of another tableau in $\mathrm{SSYT}_n(\lambda)$.

Theorem 1.4. The set $\mathrm{SSYT}_n(\lambda)$ has a unique crystal structure such that the row reading word is an isomorphism onto a full subcrystal of $\mathbb{B}_n^{\otimes|\lambda|}$.

To compute $e_i(T)$ or $f_i(T)$, use signature rule to compute $e_i(\mathfrak{row}(T))$ or $f_i(\mathfrak{row}(T))$, then split the resulting word (if nonzero) into the rows of a semistandard tableau:

$$f_2(322) = 323 \quad \rightsquigarrow \quad f_2\left(\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}\right) = \left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array}\right).$$

The crystal graph of $\mathrm{SSYT}_n(\lambda)$ is connected and it has a unique *highest weight* (an element T with $e_i(T) = 0$ for all $i \in [n-1]$), given by the tableau of shape λ with all entries in row i equal to i.

Corollary 1.5. The *Schur polynomial* $s_\lambda(x_1, x_2, \dots, x_n)$, as the character of $\mathrm{SSYT}_n(\lambda)$, is symmetric.

2 RSK correspondence

We have shown that $\mathrm{SSYT}_n(\lambda)$ is isomorphic to a full subcrystal of $\mathbb{B}_n^{\otimes|\lambda|}$.

Goal: show conversely that every full subcrystal of $\mathbb{B}_n^{\otimes m}$ is isomorphic to $\mathrm{SSYT}_n(\lambda)$ for some partition λ.

For this we need some way of turning a word $w = w_1 w_2 \cdots w_m$ into a semistandard tableau with m boxes.

Definition 2.1. Suppose T is a semistandard tableau and a is a positive integer.

Form a new tableau $T \xleftarrow{\mathsf{RSK}} a$ from T as follows:

- Add a to the end of the first row of T if the result is semistandard.
- Otherwise, there is an entry $b = T_{1j}$ in the first row of T with $a < b$.

  Choose this entry such that the column index j is as small as possible.

  Let U be the tableau formed from T by omitting the first row.

  Finally replace $b = T_{1j}$ by a and replace the remaining rows of T by $U \xleftarrow{\mathsf{RSK}} b$.

  (In this case we say that the entry b is *bumped* from column j.)

We refer to $T \xleftarrow{\mathsf{RSK}} a$ as the tableau formed by *(RSK) inserting a into T.*

Example 2.2. If $T = \emptyset$ is the empty tableau then $T \xleftarrow{\mathsf{RSK}} a = \boxed{a}$.

Example 2.3. If $T = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline\end{array}$ where $a \leq b \leq c \leq d$ then $T \xleftarrow{\mathsf{RSK}} d = \begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline\end{array}$.

Example 2.4. For positive integers $a < b < c < d$, we have

$$\begin{array}{|c|c|c|}\hline a & c & d \\ \hline\end{array} \xleftarrow{\mathsf{RSK}} b = \begin{array}{|c|c|c|}\hline a & b & d \\ \hline c \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline a & b & d \\ \hline c \\ \cline{1-1}\end{array} \xleftarrow{\mathsf{RSK}} b = \begin{array}{|c|c|c|}\hline a & b & b \\ \hline c & d \\ \cline{1-2}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline a & b & b \\ \hline c & d \\ \cline{1-2}\end{array} \xleftarrow{\mathsf{RSK}} a = \begin{array}{|c|c|c|}\hline a & a & b \\ \hline b & d \\ \cline{1-2} c \\ \cline{1-1}\end{array}.$$

Proposition 2.5. Suppose T is a semistandard tableau. Then $T \xleftarrow{\mathsf{RSK}} a$ is also semistandard.

Proof. It is clear that the rows of $T \xleftarrow{\mathsf{RSK}} a$ are weakly increasing.

We must check that the columns of this tableau are strictly increasing.

This holds trivially if $T \xleftarrow{\mathsf{RSK}} a$ is formed from T by adding a to the end of the first row.

Let U be the tableau formed from T by omitting the first row and assume a bumps a number b in column j from the first row of T.

By induction $U \xleftarrow{\mathsf{RSK}} b$ is semistandard, and if b bumps any number from the first row of U then this number will be in a column $i \leq j$, so it follows that $T \xleftarrow{\mathsf{RSK}} a$ has strictly increasing columns as needed. □

Let T be a tableau of shape λ. A pair (j, k) is a *corner box* of T if $(j, k) \in \mathsf{D}_\lambda$ but $(j+1, k), (j, k+1) \notin \mathsf{D}_\lambda$.

The corner boxes of $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$ are $(1, 3)$ and $(2, 2)$.

If T is semistandard then $T \xleftarrow{\mathsf{RSK}} a$ does not have the same shape as T.

But there is a unique corner box of $T \xleftarrow{\mathsf{RSK}} a$ we can remove to form a tableau with the same shape as T.

Definition 2.6. Suppose $w = w_1 w_2 \cdots w_m$ is a sequence of positive integers w_i.

Define a pair of tableaux $(P_{\mathsf{RSK}}(w), Q_{\mathsf{RSK}}(w))$ of the same shape as follows:

- For the empty word, define $P_{\mathsf{RSK}}(\emptyset) = Q_{\mathsf{RSK}}(\emptyset) = \emptyset$ to be the empty tableau.
- For nonempty words, let

$$P_{\mathsf{RSK}}(w) = \left(\cdots\left(\left(\emptyset \xleftarrow{\mathsf{RSK}} w_1\right) \xleftarrow{\mathsf{RSK}} w_2\right) \xleftarrow{\mathsf{RSK}} \cdots\right) \xleftarrow{\mathsf{RSK}} w_m$$

  and inductively form $Q_{\mathsf{RSK}}(w) = Q_{\mathsf{RSK}}(w_1 w_2 \cdots w_m)$ from $Q_{\mathsf{RSK}}(w_1 w_2 \cdots w_{m-1})$ by adding a box with value m to the unique position that makes $Q_{\mathsf{RSK}}(w)$ have the same shape as $P_{\mathsf{RSK}}(w)$.

The map $w \mapsto (P_{\mathsf{RSK}}(w), Q_{\mathsf{RSK}}(w))$ is called the *RSK (Robinson-Schensted-Knuth) correspondence.*

We call $P_{\mathsf{RSK}}(w)$ the *insertion tableau* and $Q_{\mathsf{RSK}}(w)$ the *recording tableau.*

Example 2.7. We have

$$\begin{aligned}
P_{\mathsf{RSK}}(45122) &= \left(\left(\left(\left(\emptyset \xleftarrow{\mathsf{RSK}} 4\right) \xleftarrow{\mathsf{RSK}} 5\right) \xleftarrow{\mathsf{RSK}} 1\right) \xleftarrow{\mathsf{RSK}} 2\right) \xleftarrow{\mathsf{RSK}} 2 \\
&= \left(\left(\left(\begin{array}{|c|}\hline 4 \\ \hline\end{array} \xleftarrow{\mathsf{RSK}} 5\right) \xleftarrow{\mathsf{RSK}} 1\right) \xleftarrow{\mathsf{RSK}} 2\right) \xleftarrow{\mathsf{RSK}} 2 \\
&= \left(\left(\begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array} \xleftarrow{\mathsf{RSK}} 1\right) \xleftarrow{\mathsf{RSK}} 2\right) \xleftarrow{\mathsf{RSK}} 2 \\
&= \left(\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \xleftarrow{\mathsf{RSK}} 2\right) \xleftarrow{\mathsf{RSK}} 2 \\
&= \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline\end{array} \xleftarrow{\mathsf{RSK}} 2 \\
&= \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}
\end{aligned}$$

and $Q_{\mathsf{RSK}}(45122) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$. Alternatively

$$\begin{aligned}
P_{\mathsf{RSK}}(41522) &= \left(\left(\left(\left(\emptyset \xleftarrow{\mathsf{RSK}} 4\right) \xleftarrow{\mathsf{RSK}} 1\right) \xleftarrow{\mathsf{RSK}} 5\right) \xleftarrow{\mathsf{RSK}} 2\right) \xleftarrow{\mathsf{RSK}} 2 \\
&= \left(\left(\left(\begin{array}{|c|}\hline 4 \\ \hline\end{array} \xleftarrow{\mathsf{RSK}} 1\right) \xleftarrow{\mathsf{RSK}} 5\right) \xleftarrow{\mathsf{RSK}} 2\right) \xleftarrow{\mathsf{RSK}} 2 \\
&= \left(\left(\begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline\end{array} \xleftarrow{\mathsf{RSK}} 5\right) \xleftarrow{\mathsf{RSK}} 2\right) \xleftarrow{\mathsf{RSK}} 2 \\
&= \left(\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \xleftarrow{\mathsf{RSK}} 2\right) \xleftarrow{\mathsf{RSK}} 2 \\
&= \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline\end{array} \xleftarrow{\mathsf{RSK}} 2 \\
&= \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}
\end{aligned}$$

and $Q_{\mathsf{RSK}}(41522) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$.

It is clear from Proposition 2.5 that $P_{\mathsf{RSK}}(w)$ is always semistandard.

A semistandard tableau with m boxes is *standard* if its entries include each of the numbers $1, 2, \dots, m$. A standard tableau must have strictly increasing rows and columns, and must contain each $i \in [m]$ in exactly one box.

A tableau T with m boxes is standard if and only if T is empty or if m appears in a corner box of T and removing this box yields a (smaller) standard tableau. This inductive characterization makes it clear that $Q_{\mathsf{RSK}}(w)$ is always a standard tableau.

We claim that the RSK correspondence is invertible. To prove this, we need another definition:

Definition 2.8. Suppose T is a semistandard tableau with a corner box (j, k).

Define column indices $\mathsf{col}_1 \geq \mathsf{col}_2 \geq \dots \geq \mathsf{col}_j = k$ and entries $\mathsf{val}_1 < \mathsf{val}_2 < \dots < \mathsf{val}_j = T_{jk}$ as follows:

- Set $\mathsf{col}_j = k$ and $\mathsf{val}_j = T_{jk}$.
- For each $1 \leq i < j$, define val_i to be the largest entry in row i of T with $\mathsf{val}_i < \mathsf{val}_{i+1}$.

  Then define col_i to be the largest column containing this entry in row i of T.

Now set $a = \mathsf{val}_1$ and form the tableau U from T as follows:

- First remove box (j, k).

– Then replace the entry val_i in position (i, col_i) of T by val_{i+1} for each $1 \leq i < j$.

In this event we write $T \xrightarrow{(j,k)} (U, a)$.

More generally, if $bcd \cdots z$ is any word and $T \xrightarrow{(j,k)} (U, a)$ then we write $(T, bcd \cdots z) \xrightarrow{(j,k)} (U, abcd \cdots z)$.

We refer to the operations $T \xrightarrow{(j,k)} (U, a)$ or $(T, w) \xrightarrow{(j,k)} (U, aw)$ as *unbumping* corner box (j, k).

Example 2.9. If $T = \begin{ytableau} a & b & c & d \end{ytableau}$ then $T \xrightarrow{(1,4)} \left(\begin{ytableau} a & b & c \end{ytableau}, d\right)$.

Example 2.10. If $T = \begin{ytableau} a & a & b \\ b & d \\ c \end{ytableau}$ where $a < b < c < d$ and $(j, k) = (3, 1)$ then

$$\mathsf{col}_3 = 1 \leq \mathsf{col}_2 = 1 < \mathsf{col}_1 = 2 \qquad \text{and} \qquad \mathsf{val}_3 = c > \mathsf{val}_2 = b > \mathsf{val}_1 = a$$

so we have $T \xrightarrow{(3,1)} \left(\begin{ytableau} a & b & b \\ c & d \end{ytableau}, a\right)$. On the other hand

$$T \xrightarrow{(2,2)} \left(\begin{ytableau} a & a & d \\ b \\ c \end{ytableau}, b\right) \xrightarrow{(3,1)} \left(\begin{ytableau} a & b & d \\ c \end{ytableau}, ab\right) \xrightarrow{(1,3)} \left(\begin{ytableau} a & b \\ c \end{ytableau}, dab\right)$$
$$\xrightarrow{(2,1)} \left(\begin{ytableau} a & c \end{ytableau}, bdab\right) \xrightarrow{(1,2)} \left(\begin{ytableau} a \end{ytableau}, cbdab\right) \xrightarrow{(1,1)} \left(\emptyset, acbdab\right).$$

Lemma 2.11. Suppose T is a semistandard tableau with corner box (j, k).

If $T \xrightarrow{(j,k)} (U, a)$ then U is semistandard and $T = U \xleftarrow{\mathsf{RSK}} a$.

Proof. It is clear that the rows of U are weakly increasing. To see that the columns of this tableau are strictly increasing, it suffices to check (using the notation of Definition 2.8) that if $\mathsf{col}_{i+1} < \mathsf{col}_i$ then box $(i + 1, \mathsf{col}_i)$ of T is either unoccupied or contains an entry strictly larger than val_{i+1}.

The claim that $T \xrightarrow{(j,k)} (U, a)$ implies $T = U \xleftarrow{\mathsf{RSK}} a$ is easy to verify. □

Theorem 2.12. The RSK correspondence $w \mapsto (P_{\mathsf{RSK}}(w), Q_{\mathsf{RSK}}(w))$ is a bijection from words to pairs (P, Q) of semistandard tableaux of the same shape with Q standard.

If w has length m then $P_{\mathsf{RSK}}(w)$ and $Q_{\mathsf{RSK}}(w)$ both have m boxes, and $\mathbf{wt}(w) = \mathbf{wt}(P_{\mathsf{RSK}}(w))$.

Proof. Suppose (P, Q) is a pair of semistandard tableaux of the same shape with Q standard.

Assume each tableau has m boxes. Write $Q^{-1}(i)$ for the position of the box containing $i \in [m]$ in Q.

Define P_{i-1} and w_i for $i \in [m]$ such that $P_i \xrightarrow{Q^{-1}(i)} (P_{i-1}, w_i)$ where $P_m = P$.

Then $Q^{-1}(i)$ is always a corner box of P_i and $P_0 = \emptyset$, so we have

$$P \xrightarrow{Q^{-1}(m)} (P_m, w_m) \xrightarrow{Q^{-1}(m-1)} (P_{m-1}, w_{m-1}w_m) \xrightarrow{Q^{-1}(m-2)} \cdots \xrightarrow{Q^{-1}(1)} (\emptyset, w_1 w_2 \cdots w_m).$$

By the previous lemma $(P, Q) \mapsto w_1 w_2 \cdots w_m$ and $w \mapsto (P_{\mathsf{RSK}}(w), Q_{\mathsf{RSK}}(w))$ are inverse maps. □

Corollary 2.13. The number of pairs (P, Q) of semistandard tableaux of the same shape with Q standard, such that P has m boxes with all entries in $\{1, 2, \dots, n\}$, is n^m.

Proof. Such pairs are in bijection via RSK with words $w_1 w_2 \cdots w_m$ where each $w_i \in \{1, 2, \dots, n\}$. □

Corollary 2.14. The number of pairs (P, Q) of standard tableaux of the same shape with n boxes is $n!$.

Proof. Such pairs are in bijection via RSK with words $w_1 w_2 \cdots w_n$ where $\{w_1, \dots, w_n\} = \{1, \dots, n\}$. □

Returning to crystals, we have an important complement to the previous theorem.

Here, we will omit $\otimes$ symbols and write elements of $\mathbb{B}_n^{\otimes m}$ as words $w = w_1 w_2 \cdots w_m$.

Write $\bigsqcup$ for disjoint union. The disjoint union of an arbitrary collection of crystals is itself a crystal.

Theorem 2.15. The map $w \mapsto P_{\mathsf{RSK}}(w)$ is a crystal morphism $\mathbb{B}_n^{\otimes m} \to \bigsqcup_\lambda \mathrm{SSYT}_n(\lambda)$.

More strongly, suppose $w = w_1 w_2 \cdots w_m \in \mathbb{B}_n^{\otimes m}$ and $i \in [n-1]$. Then:

(a) If $e_i(w) = 0$ then $e_i(P_{\mathsf{RSK}}(w)) = 0$, while if $e_i(w) \neq 0$ then

$$e_i(P_{\mathsf{RSK}}(w)) = P_{\mathsf{RSK}}(e_i(w)) \qquad \text{and} \qquad Q_{\mathsf{RSK}}(w) = Q_{\mathsf{RSK}}(e_i(w)).$$

(b) If $f_i(w) = 0$ then $f_i(P_{\mathsf{RSK}}(w)) = 0$, while if $f_i(w) \neq 0$ then

$$f_i(P_{\mathsf{RSK}}(w)) = P_{\mathsf{RSK}}(f_i(w)) \qquad \text{and} \qquad Q_{\mathsf{RSK}}(w) = Q_{\mathsf{RSK}}(f_i(w)).$$

The slightly subtle details of the proof of this theorem will be carried out in our first homework assignment.

We note some consequences of this result.

Corollary 2.16. The full subcrystals of $\mathbb{B}_n^{\otimes m}$ are the subsets of the form $\{w : Q_{\mathsf{RSK}}(w) = T\}$ where T ranges over all standard tableaux with m boxes and entries in $\{1, 2, \dots, n\}$.

Proof. The theorem shows that $\{w : Q_{\mathsf{RSK}}(w) = T\}$ is at least a union of full subcrystals of $\mathbb{B}_n^{\otimes m}$, and that $w \mapsto P_{\mathsf{RSK}}(w)$ defines a crystal isomorphism $\{w : Q_{\mathsf{RSK}}(w) = T\} \to \mathrm{SSYT}_n(\lambda)$ where λ is the shape of T. Since $\mathrm{SSYT}_n(\lambda)$ is a connected crystal, $\{w : Q_{\mathsf{RSK}}(w) = T\}$ must be a full subcrystal. □

We define a crystal to be *normal* if each of its full subcrystals is isomorphic to a full subcrystal of some tensor power of the standard crystal $\mathbb{B}_n^{\otimes m}$.

Corollary 2.17. The character of a finite normal crystal is *Schur positive*, that is, a finite linear combination of Schur polynomials with nonnegative integer coefficients.

Proof. Every full subcrystal of a normal crystal is isomorphic to $\mathrm{SSYT}_n(\lambda)$ for some λ. □

Corollary 2.18. A connected normal crystal has a unique highest weight element. The weight of this element is a partition (after omitting trailing zeros) and determines the crystal up to isomorphism.

Proof. This is true for $\mathrm{SSYT}_n(\lambda)$, whose unique highest weight has weight λ, and therefore also for any connected normal crystal. □

Lemma 2.19. The tensor product of two normal crystals is again normal.

Proof. The tensor product of subcrystals of $\mathbb{B}_n^{\otimes p}$ and $\mathbb{B}_n^{\otimes q}$ is isomorphic to a subcrystal of $\mathbb{B}_n^{\otimes (p+q)}$. □

Finally, we recover the second nontrivial theorem about Schur polynomials from our first lecture.

Corollary 2.20. For each triple of partitions λ, μ, ν, there are nonnegative integer coefficients $c^\nu_{\lambda\mu}$ such that $s_\lambda(x_1, x_2, \dots, x_n)s_\mu(x_1, x_2, \dots, x_n) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(x_1, x_2, \dots, x_n)$.

Proof. The product $s_\lambda(x_1, x_2, \dots, x_n)s_\mu(x_1, x_2, \dots, x_n)$ is the character of $\mathrm{SSYT}_n(\lambda) \otimes \mathrm{SSYT}_n(\mu)$, which is a normal crystal. We therefore have $s_\lambda(x_1, x_2, \dots, x_n)s_\mu(x_1, x_2, \dots, x_n) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(x_1, x_2, \dots, x_n)$ where $c^\nu_{\lambda\mu}$ is the number of highest weight elements in $\mathrm{SSYT}_n(\lambda) \otimes \mathrm{SSYT}_n(\mu)$ with weight ν. □