1 Last time: normal crystals

Fix a positive integer n as usual. Here is a useful fact:

Observation 1.1. Schur polynomials $s_\lambda(x_1,\dots,x_n) = \sum_{T\in\mathrm{SSYT}_n(\lambda)} x^{\mathbf{wt}(\lambda)}$ are linearly independent.

Proof. Let $\prec$ be the partial order on $\mathbb{Z}^n$ with $\alpha + \mathbf{e}_i - \mathbf{e}_{i+1} \prec \alpha$ for all $i \in [n-1]$.

Then $s_\lambda(x_1,\dots,x_n) = x^\lambda + ($ monomials indexed by higher order terms under $\prec$ $)$.

So we get linear independence from a standard triangularity argument. $\square$

Unimportant terminology: from a category, one forms a *full subcategory* by taking a subset of objects but keeping all morphisms between these objects that were in the original category.

So for example, finite dimensional complex vector spaces form a full subcategory of the category of all complex vector spaces with linear maps as morphisms. On the other hand, topological vector spaces form a subcategory that is not full, since we only keep continuous linear maps as morphisms.

Last time we showed that our category of crystals has a full subcategory whose objects we call *normal crystals*, characterized by the following properties:

- The standard crystal $\mathbb{B}_n$ is a normal crystal.
- Tensor products of normal crystals are normal.
- Full subcrystals of normal crystals are normal.
- Disjoint unions of normal crystals are normal.
- Each connected normal crystal is isomorphic to $\mathrm{SSYT}_n(\lambda)$ for some partition λ with $\leq n$ parts.
- None of the crystals $\mathrm{SSYT}_n(\lambda)$ are isomorphic since each has a distinct highest weight, namely λ.
- Each connected normal crystal has a unique highest weight which determines its isomorphism class.
- Since $\mathrm{ch}(\mathrm{SSYT}_n(\lambda)) = s_\lambda(x_1,\dots,x_n)$, the character of any finite normal crystal is *Schur positive*.
- Two finite normal crystals are isomorphic if and only if they have the same character.

Our goal going forward is to replicate the construction of this category for any *Cartan type*.

For this, we will need to introduce more general notions of crystals.

2 Aside: Littlewood-Richardson coefficients

An application of all this is a concrete interpretation of the *Littlewood-Richardson coefficients* $c^\nu_{\lambda\mu} \in \mathbb{N}$.

Let λ and μ be partitions. The tensor product $\mathrm{SSYT}_n(\lambda)\otimes\mathrm{SSYT}_n(\mu)$ is a normal crystal, so there are unique coefficients $c^\nu_{\lambda\mu} \in \mathbb{N}$ indexed by partitions ν such that

$$\mathrm{SSYT}_n(\lambda)\otimes\mathrm{SSYT}_n(\mu) \cong \bigsqcup_\nu \underbrace{\mathrm{SSYT}_n(\nu)\sqcup\mathrm{SSYT}_n(\nu)\sqcup\cdots\sqcup\mathrm{SSYT}_n(\nu)}_{c^\nu_{\lambda\mu}\text{ times}}.$$

(In this disjoint union, we skip any indices ν with $c^\nu_{\lambda\mu} = 0$.)

Specifically, $c^\nu_{\lambda\mu}$ is the number of highest weight elements in $\mathrm{SSYT}_n(\lambda)\otimes\mathrm{SSYT}_n(\mu)$ with weight ν.

Since $\mathrm{ch}(\mathcal{B}\otimes\mathcal{C}) = \mathrm{ch}(\mathcal{B})\mathrm{ch}(\mathcal{C})$ and $\mathrm{ch}(\mathcal{B}\sqcup\mathcal{C}) = \mathrm{ch}(\mathcal{B}) + \mathrm{ch}(\mathcal{C})$, taking characters gives

$$s_\lambda(x_1,\dots,x_n)s_\mu(x_1,\dots,x_n) = \sum_\nu c^\nu_{\lambda\mu}s_\nu(x_1,\dots,x_n) \in \mathbb{N}\text{-span}\Big\{s_\nu(x_1,\dots,x_n) : \nu\vdash|\lambda|+|\nu|\Big\}. \tag{2.1}$$

Here, "$\nu \vdash k$" means $|\nu| = k$.

The way we have defined $c^\nu_{\lambda\mu}$ has an implicit dependence n. We can remove this dependence as follows.

Definition 2.1. Given partitions λ, μ, ν, define $c^\nu_{\lambda\mu} \in \mathbb{N}$ to be the number of pairs (T', T'') of semistandard tableaux of shape λ and μ, respectively, such that $P_{\mathsf{RSK}}(\mathfrak{row}(T')\mathfrak{row}(T''))$ is the unique tableau of shape ν with all entries in row i equal to i. One calls these integers $c^\nu_{\lambda\mu}$ *Littlewood-Richardson coefficients*.

If ν has $\ell(\nu)$ parts then the set of entries in T' or T'' for any such pair must be exactly $\{1, 2, 3, \dots, \ell(\nu)\}$. Therefore, by results last time, $c^\nu_{\lambda\mu}$ is exactly the number of highest weight elements in the tensor product $\mathrm{SSYT}_n(\lambda) \otimes \mathrm{SSYT}_n(\mu)$ with weight ν whenever $n \geq \max\{\ell(\lambda), \ell(\mu), \ell(\nu)\}$.

Since $s_\lambda(x_1, \dots, x_n) = 0$ if $n < \ell(\lambda)$, it follows that (2.1) still holds for our new definition of $c^\nu_{\lambda\mu}$.

Our definition of $c^\nu_{\lambda\mu}$ is now independent of n. This means that we can extend (2.1) to an identity for the *Schur functions* $s_\lambda := \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathbf{wt}(T)} = \lim_{n\to\infty} s_\lambda(x_1, \dots, x_n)$.

Here $\mathrm{SSYT}(\lambda) := \bigcup_{n \geq 1} \mathrm{SSYT}_n(\lambda)$. Schur functions are formal power series.

Theorem 2.2. For any partitions λ and μ, it holds that $s_\lambda s_\mu = \sum_{\nu \vdash |\lambda|+|\mu|} c^\nu_{\lambda\mu} s_\nu$.

We can always specialize a power series by setting $x_{n+1} = x_{n+2} = \dots = 0$, so this generalizes (2.1).

3 Cartan types

In this course, a *Cartan type* consists of a pair (Φ, Λ) where Φ is a root system and Λ is a weight lattice.

Informally, a root system Φ is a finite set of vectors in a real vector space V, and a weight lattice is a finitely generated abelian subgroup Λ of V that spans V, modulo a few other conditions.

The vector space V has an inner product $\langle \cdot, \cdot \rangle$ which is the standard form when $V = \mathbb{R}^n$.

The set of roots Φ has a subset of positive roots Φ^+ and simple roots $\{\alpha_i : i \in I\}$.

The simple roots determine a set of dominant weights $\Lambda^+ \subset \Lambda$ and fundamental weights $\{\varpi_i : i \in I\}$.

The simple roots and fundamental weights have the same indexing set I, whose size is the *rank*.

Before discussing the precise axioms, let's present two main examples.

Example 3.1 (Cartan type A_{n-1}, $\mathrm{GL}(n)$ version)**.**

For this Cartan type we take $V = \mathbb{R}^n$, $\Lambda = \mathbb{Z}^n$, and $\Phi = \{\mathbf{e}_i - \mathbf{e}_j : i, j \in [n],\ i \neq j\}$.

The positive roots are $\Phi^+ = \{\mathbf{e}_i - \mathbf{e}_j : 1 \leq i < j \leq n\}$. The simple roots are $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i \in [n-1]$.

The dominant weights are the vectors $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

The fundamental weights are $\varpi_i = \mathbf{e}_1 + \mathbf{e}_2 + \dots + \mathbf{e}_i$ for $i \in [n-1]$.

We have been implicitly working with this Cartan type in the lectures so far.

There is one other "type A" Cartan type:

Example 3.2 (Cartan type A_{n-1}, $\mathrm{SL}(n)$ version)**.**

For this Cartan type we take $V = \mathbb{R}^n/\mathbb{R}\text{-span}\{\mathbf{e}_1 + \mathbf{e}_2 + \dots + \mathbf{e}_n\}$ to be a quotient vector space.

The root system Φ and weight lattice Λ are the images of $\{\mathbf{e}_i - \mathbf{e}_j :\ i \neq j\}$ and $\mathbb{Z}^n$ in this quotient.

The positive roots, simple roots, dominant weights, and fundamental weights are also just the images in this quotient of their counterparts for the $\mathrm{GL}(n)$-case.

We now give the definitions in general, before presenting types B, C, and D.

Let V be a real vector space with an inner product $\langle\cdot,\cdot\rangle$. An *inner product* is a symmetric positive definite bilinear form. We can usually think of this as the standard form on $\mathbb{R}^n$.

Each nonzero $\alpha \in V$ determines a linear map $r_\alpha : V \to V$ by the formula

$$r_\alpha(x) = x - 2\tfrac{\langle x,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha \qquad \text{for } x \in V.$$

You can check that $(r_\alpha)^2 = 1$ and $r_\alpha(\alpha) = -\alpha$. Given any nonzero $\alpha \in V$, let $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$.

Definition 3.3. A *root system* in V is a set of nonzero vectors $\Phi \subset V \setminus \{0\}$ such that

(1) We have $r_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

(2) We have $\langle \alpha, \beta^\vee\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

(3) If $\beta \in \Phi$ is a multiple of $\alpha \in \Phi$ then $\beta = \pm\alpha$.

Assume Φ is a root system in V. Then $\Phi = -\Phi$ since $r_\alpha(\alpha) = -\alpha$.

Elements of Φ are often called *roots*. Elements of $\Phi^\vee := \{\alpha^\vee : \alpha \in \Phi\}$ are called coroots.

In both type A Cartan types, we have $\Phi = \Phi^\vee$.

If we can write $\Phi = \Phi_1 \sqcup \Phi_2$ where each Φ_i is a nonempty set and $\langle \alpha_1, \alpha_2\rangle = 0$ for all $\alpha_i \in \Phi_i$, then each of the subsets Φ_i is itself a root system and we say that Φ is *reducible*.

When this is not possible, Φ is *irreducible* or *simple*. Every root system is a disjoint union of a finite number of irreducible root systems. These have a fairly tame classification, which will be covered in any introductory course on Lie algebras.

A root system is *simply-laced* if every root has the same length. This is always the case in type A.

The root system Φ is *semisimple* if $V = \mathbb{R}\Phi$, i.e., if V is spanned by Φ.

This occurs in the Cartan type for $\mathrm{SL}(n)$ but not $\mathrm{GL}(n)$.

Definition 3.4. Let Φ be a root system in V.

A *weight lattice* Λ for Φ is a finitely generated abelian subgroup of V such that:

(1) We have $V = \mathbb{R}\Lambda$, i.e., the $\mathbb{R}$-span of Λ is V.

(2) The root system Φ is a subset of Λ.

(3) We have $\langle \lambda, \alpha^\vee\rangle \in \mathbb{Z}$ for all $\lambda \in \Lambda$ and $\alpha \in \Phi$.

These conditions trivially hold for $\Lambda = \mathbb{Z}^n$ in the $\mathrm{GL}(n)$ Cartan type.

Let Λ be a weight lattice for the root system Φ in V. Elements of Λ are called *weights*.

Choose a codimension one subspace $H \subset V$ that does not intersect Φ.

Equivalently, choose a vector ρ (orthogonal to H) that is not orthogonal to any root in Φ.

Whether $\langle \alpha, \rho\rangle$ is positive or negative tells us which side of H the root α is on.

The set of *positive roots* in Φ is then $\Phi^+ = \{\alpha : \langle\alpha,\rho\rangle > 0\}$. The *negative roots* are $\Phi^- = -\Phi^+$.

In the $\mathrm{GL}(n)$ Cartan type, the positive roots were constructed using the vector $\rho = (n-1, \dots, 2, 1, 0)$.

A root $\alpha \in \Phi^+$ is *simple* if it cannot be written as the sum of two positive roots.

Let $\{\alpha_i : i \in I\}$ be the set of simple roots in Φ^+. We usually identify the indexing set I with set $[n]$.

We recall some nontrivial facts (omitting the proofs):

- The simple roots are linearly independent.
- If $i, j \in I$ then $i \neq j$ then $\langle \alpha_i, \alpha_j \rangle \leq 0$.
- Each positive root is a linear combination of simple roots with nonnegative coefficients.

Also define $s_i = r_{\alpha_i}$ for $i \in I$. Then:

- Each reflection s_i maps $\alpha_i \mapsto -\alpha_i$ and acts as a permutation of $\Phi^+ \setminus \{\alpha_i\}$.
- The set of reflections $\{s_i : i \in I\}$ is the simple generating set of a Coxeter group W.

In type A_{n-1}, the group W is isomorphic (though not obviously) to the symmetric group S_n.

A weight $\lambda \in \Lambda$ is *dominant* if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $i \in I$.

A dominant weight is *strictly dominant* if these inner products are always positive.

Let Λ^+ be the set of dominant weights in Λ.

One can show that Λ^+ always contains the *Weyl vector* $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$.

In type A_{n-1} we have $\alpha_i^\vee = \alpha_i$ so $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is dominant when its entries are weakly decreasing.

We make one final choice: let $\{\varpi_i : i \in I\}$ be a set of vectors in V such that

$$\langle \varpi_i, \alpha_j^\vee \rangle = \begin{cases} 1 & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

We call these elements *fundamental weights*.

If Φ is semisimple then the weights are the unique basis dual to the basis of simple coroots.

Otherwise, we have some flexibility in our choice of fundamental weights, which need not belong to Λ.

Let $\Lambda_{\mathrm{sc}} = \mathbb{Z}\text{-span}\{\varpi_i : i \in I\}$ and $\Lambda_{\mathrm{root}} = \mathbb{Z}\text{-span}\{\alpha_i : i \in I\}$.

If $\Lambda = \Lambda_{\mathrm{root}}$ then we say that the weight lattice is of *adjoint type*.

If $\Lambda = \Lambda_{\mathrm{sc}}$ then we say that the weight lattice is of *simply-connected type*.

The Cartan type for $\mathrm{GL}(n)$ is of simply-connected type while the one for $\mathrm{SL}(n)$ is of adjoint type.

There are five other families of classical Cartan types.

Example 3.5 (Cartan type B_n, $\mathrm{spin}(2n+1)$ version)**.**

For this Cartan type we take $V = \mathbb{R}^n$ and $\Phi = \{\pm \mathbf{e}_i \pm \mathbf{e}_j : 1 \leq i < j \leq n\} \sqcup \{\pm \mathbf{e}_i : i \in [n]\}$.

The positive roots are $\Phi^+ = \{\mathbf{e}_i \pm \mathbf{e}_j : 1 \leq i < j \leq n\} \sqcup \{\mathbf{e}_i : i \in [n]\}$.

The weight lattice Λ is the subset of vectors $\lambda \in \mathbb{Q}^n$ with $2\lambda_i \in \mathbb{Z}$ and $2\lambda_1 \equiv 2\lambda_2 \equiv \cdots \equiv 2\lambda_n \pmod 2$.

The dominant weights are the weights whose entries are weakly decreasing.

The simple roots are $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i \in [n-1]$ along with $\alpha_n = \mathbf{e}_n$.

The fundamental weights are $\varpi_i = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_i$ for $i \in [n-1]$ along with $\varpi_n = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n)$.

In this example the root system is not simply-laced but the weight lattice is of simply-connected type.

Example 3.6 (Cartan type B_n, $\mathrm{SO}(2n+1)$ version)**.**

Again let $V = \mathbb{R}^n$. We take the same root system Φ as in the previous example but set $\Lambda = \mathbb{Z}^n$.

The simple roots and fundamental weights are the same as in the $\mathrm{spin}(2n+1)$ Cartan type.

Example 3.7 (Cartan type C_n)**.**

For this Cartan type we take $V = \mathbb{R}^n$ and $\Phi = \{\pm\mathbf{e}_i \pm \mathbf{e}_j : 1 \le i < j \le n\} \sqcup \{\pm 2\mathbf{e}_i : i \in [n]\}$.

The positive roots are $\Phi^+ = \{\mathbf{e}_i \pm \mathbf{e}_j : 1 \le i < j \le n\} \sqcup \{2\mathbf{e}_i : i \in [n]\}$.

The weight lattice is $\Lambda = \mathbb{Z}^n$.

The dominant weights are the weights whose entries are weakly decreasing.

The simple roots are $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i \in [n-1]$ along with $\alpha_n = 2\mathbf{e}_n$.

The fundamental weights are $\varpi_i = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_i$ for $i \in [n]$.

In this example the root system is not simply-laced.

Example 3.8 (Cartan type D_n, $\mathrm{spin}(2n)$ version)**.**

For this Cartan type we take $V = \mathbb{R}^n$ and $\Phi = \{\pm\mathbf{e}_i \pm \mathbf{e}_j : 1 \le i < j \le n\} = \Phi^\vee$.

The positive roots are $\Phi^+ = \{\mathbf{e}_i \pm \mathbf{e}_j : 1 \le i < j \le n\}$.

The weight lattice is the same as the $\mathrm{spin}(2n+1)$ weight lattice.

The dominant weights are the weights λ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{n-1} \ge |\lambda_n|$.

The simple roots are $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i \in [n-1]$ along with $\alpha_n = \mathbf{e}_{n-1} + \mathbf{e}_n$.

The fundamental weights are $\varpi_i = \mathbf{e}_1 + \cdots + \mathbf{e}_i$ for $i \in [n-2]$ along with

$$\varpi_{n-1} = \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n-1} - \mathbf{e}_n) \qquad \text{and} \qquad \varpi_n = \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n-1} + \mathbf{e}_n).$$

This example is simply-laced and of simply-connected type.

Example 3.9 (Cartan type D_n, $\mathrm{SO}(2n)$ version)**.**

Again let $V = \mathbb{R}^n$. We take the same root system Φ as in the previous example but set $\Lambda = \mathbb{Z}^n$.

The simple roots and fundamental weights are the same as in the $\mathrm{spin}(2n)$ Cartan type.

Example 3.10 (Exceptional types)**.**

There are a finite list of exceptional Cartan types that we refer to as types G_2, F_4, E_6, E_7, and E_8.

4 Kashiwara crystals

Crystals for arbitrary Cartan types are often called *Kashiwara crystals*. The theory of these objects was first developed in separate papers of Kashiwara and Lusztig that first appeared in 1990.

We consider the set $\mathbb{Z} \sqcup \{-\infty\}$ to be totally ordered with $-\infty < n$ for all $n \in \mathbb{Z}$.

In the following definition, we also let $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$.

Definition 4.1. Fix a root system Φ with simple roots $\{\alpha_i : i \in I\}$ and weight lattice Λ.

A *(Kashiwara) crystal* of type (Φ, Λ) is a nonempty set $\mathcal{B}$ with maps

$$\mathbf{wt} : \mathcal{B} \to \Lambda, \qquad e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \text{ for } i \in I, \qquad \text{and} \qquad \varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\} \text{ for } i \in I,$$

where $0 \notin \mathcal{B}$ is an auxiliary element, such that for $i \in I$ the following properties hold:

(1) If $x, y \in \mathcal{B}$ then $e_i(x) = y$ if and only if $f_i(y) = x$, in which case

$$\mathbf{wt}(y) = \mathbf{wt}(x) + \alpha_i, \qquad \varepsilon_i(y) = \varepsilon_i(x) - 1, \qquad \text{and} \qquad \varphi_i(y) = \varphi_i(x) + 1.$$

(2) If $x \in \mathcal{B}$ then $\varphi_i(x) = \varepsilon_i(x) + \langle \mathbf{wt}(x), \alpha_i^\vee \rangle$. In particular, if $\varphi_i(x) = -\infty$ then $\varepsilon_i(x) = -\infty$.
Also, if $\varphi_i(x) = \varepsilon_i(x) = -\infty$ then we require $e_i(x) = f_i(x) = 0$.

The function $\mathbf{wt}$ is the *weight map* of $\mathcal{B}$.

The maps e_i and f_i are the *raising* and *lowering crystal operators*.

The maps ε_i and φ_i are the *string lengths*.

If φ_i and ε_i do not take the value $-\infty$, then the crystal is of *finite type*. If

$$\varepsilon_i(b) = \max\{k \geq 0 : e_i^k(b) \neq 0\} \qquad \text{and} \qquad \varphi_i(b) = \max\{k \geq 0 : f_i^k(b) \neq 0\}$$

for all $i \in I$, then the crystal is *seminormal*.

Any seminormal crystal is of finite type, but finite type crystals may have an infinite number of elements.

The crystals we considered in the first three lectures were all seminormal of Cartan type $\mathrm{GL}(n)$.