1 Last time: root systems and weight lattices

For us, a *Cartan type* refers to a pair (Φ, Λ) where Φ is a root system and Λ is a weight lattice.

A *root system* Φ is a finite set of vectors α in a real inner product space V that is invariant under the reflections $r_\alpha : x \mapsto x - 2\frac{\langle x,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$ for all $\alpha \in \Phi$, with $\langle \alpha, \beta^\vee\rangle \in \mathbb{Z}$ and $\mathbb{Z}\alpha \cap \Phi = \{\pm\alpha\}$ for all $\alpha, \beta \in \Phi$.

A root system comes with sets of *positive roots* Φ^+, *negative roots* Φ^-, and *simple roots* $\{\alpha_i : i \in I\}$

Every root system is a disjoint union of mutually orthogonal irreducible root systems, which can belong to four infinite families of classical types A_n, B_n, C_n, D_n or five exceptional types E_6, E_7, E_8, F_4, G_2.

A *weight lattice* Λ for a root system Φ in V is a finitely generated abelian group of vectors that spans V, contains Φ, and satisfies $\langle \lambda, \alpha^\vee\rangle \in \mathbb{Z}$ for all $\lambda \in \Lambda$ and $\alpha \in \Phi$, where $\alpha^\vee := \frac{2\alpha}{\langle \alpha,\alpha\rangle}$ is called a *coroot*.

A weight lattice comes with a set of *dominant weights* $\Lambda^+ = \{\lambda \in \Lambda : \langle \lambda, \alpha_i^\vee\rangle \geq 0 \text{ for all } i \in I\}$.

We also choose *fundamental weights* $\{\varpi_i : i \in I\}$ that satisfy $\langle \varpi_i, \alpha_j^\vee\rangle = \delta_{ij}$.

The root system Φ is *semisimple* if $V = \mathbb{R}\Phi$. The weight lattice Λ is of *adjoint type* if generated by the simple roots, and of *simply-connected type* if generated by the fundamental weights.

Fix a positive integer n. Here is the standard example to keep in mind:

Example 1.1 (Cartan type A_{n-1}, $\mathrm{GL}(n)$ version)**.**

For this Cartan type we take $V = \mathbb{R}^n$, $\Lambda = \mathbb{Z}^n$, and $\Phi = \{\mathbf{e}_i - \mathbf{e}_j : i, j \in [n],\ i \neq j\}$.

Positive roots are $\Phi^+ = \{\mathbf{e}_i - \mathbf{e}_j : 1 \leq i < j \leq n\}$. Simple roots are $\alpha_i := \mathbf{e}_i - \mathbf{e}_{i+1} = \alpha_i^\vee$ for $i \in [n-1]$.

The dominant weights are the vectors $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

The fundamental weights are $\varpi_i := \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_i$ for $i \in [n-1]$. Note that $\langle \varpi_i, \alpha_j^\vee\rangle = \delta_{ij} \in \{0, 1\}$.

2 Properties of crystals for general Cartan types

Fix a root system Φ with simple roots $\{\alpha_i : i \in I\}$ and weight lattice Λ.

The set $\mathbb{Z} \sqcup \{-\infty\}$ is totally ordered with $-\infty < n$ for all $n \in \mathbb{Z}$. Let $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$.

Here is the general definition of a crystal from the end of last lecture:

Definition 2.1. A *(Kashiwara) crystal* of type (Φ, Λ) is a nonempty set $\mathcal{B}$ with maps

$$\mathbf{wt} : \mathcal{B} \to \Lambda, \qquad e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \text{ for } i \in I, \qquad \text{and} \qquad \varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\} \text{ for } i \in I,$$

where $0 \notin \mathcal{B}$ is an auxiliary element, such that for each $i \in I$ the following properties hold:

(1) If $x, y \in \mathcal{B}$ then $e_i(x) = y$ if and only if $f_i(y) = x$, in which case

$$\mathbf{wt}(y) = \mathbf{wt}(x) + \alpha_i, \qquad \varepsilon_i(y) = \varepsilon_i(x) - 1, \qquad \text{and} \qquad \varphi_i(y) = \varphi_i(x) + 1.$$

(2) If $x \in \mathcal{B}$ then $\varphi_i(x) = \varepsilon_i(x) + \langle \mathbf{wt}(x), \alpha_i^\vee\rangle$. In particular, if $\varphi_i(x) = -\infty$ then $\varepsilon_i(x) = -\infty$.

Also, if $\varphi_i(x) = \varepsilon_i(x) = -\infty$ then we require $e_i(x) = f_i(x) = 0$.

Refer to $\mathbf{wt}$ as the *weight map*; to e_i, f_i as the *crystal operators*; and to ε_i, φ_i as the *string lengths*.

Important special cases: if φ_i and ε_i do not take the value $-\infty$, then the crystal is of *finite type*. If

$$\varepsilon_i(b) = \max\{k \geq 0 : e_i^k(b) \neq 0\} \qquad \text{and} \qquad \varphi_i(b) = \max\{k \geq 0 : f_i^k(b) \neq 0\}$$

for all $i \in I$ and $b \in \mathcal{B}$, then the crystal is *seminormal*. Seminormal implies finite type.

Example 2.2. For each $\lambda \in \Lambda$, there is a crystal $\mathcal{T}_\lambda = \{t_\lambda\}$ consisting a single element with $\mathbf{wt}(t_\lambda) = \lambda$ and $f_i(t_\lambda) = e_i(\lambda) = 0$ and $\varphi_i(t_\lambda) = \varepsilon_i(t_\lambda) = -\infty$ for all $i \in I$. This finite crystal is not of finite type.

By now we have some intuition for crystals in the seminormal case of the $\mathrm{GL}(n)$ Cartan type.

Today we will discuss how various properties of those crystals extend to other types.

Proposition 2.3. Assume the Cartan type (Φ, Λ) is *semisimple* in the sense that $V = \mathbb{R}\Phi$.

Let $\mathcal{B}$ be a finite type crystal for (Φ, Λ). If $x \in \mathcal{B}$ then $\mathbf{wt}(x) = \sum_{i\in I} (\varphi_i(x) - \varepsilon_i(x))\,\varpi_i$.

Proof. Since $\mathcal{B}$ is of finite type, the crystal axioms imply that for each index $j \in I$ we have

$$\left\langle \mathbf{wt}(x) - \sum_{i\in I} (\varphi_i(x) - \varepsilon_i(x))\,\varpi_i, \alpha_j^\vee \right\rangle = \langle \mathbf{wt}(x), \alpha_j^\vee\rangle - \varphi_j(x) + \varepsilon_j(x) = 0.$$

The simple coroots are a basis for V since we are in the semisimple case.

Therefore, as the inner product for V is positive definite, if $\langle v, \alpha_j^\vee\rangle = 0$ for all $j \in I$ then $v = 0$. $\square$

For general crystals $\mathcal{B}$ we have some terminology that directly carries over from the type A case:

- The *crystal graph* is the directed graph with edges $x \xrightarrow{i} y$ for $x, y \in \mathcal{B}$ and $i \in I$ with $f_i(x) = y$.
- A weakly connected component of the crystal graph is a *full subcrystal*.
- A crystal with just one full subcrystal is *connected*.
- Each crystal is a disjoint union of full subcrystals, which we call the crystal's *connected components*.
- An element $u \in \mathcal{B}$ is a *highest weight element* if $e_i(u) = 0$ for all $i \in I$.
- The value of $\mathbf{wt}(u) \in \Lambda$ for a highest weight element is a *highest weight*.

The weight lattice Λ is partially ordered by the relation $\preceq$ that has $\mu \preceq \lambda$ if $\lambda - \mu$ is a linear combination of simple roots α_i with nonnegative coefficients.

Lemma 2.4. Suppose $u \in \mathcal{B}$ is an element such that $\mathbf{wt}(u)$ is maximal in the set $\{\mathbf{wt}(b) : b \in \mathcal{B}\}$ with respect to $\preceq$. Then u is a highest weight element.

Proof. This holds since if $e_i(u) \neq 0$ then $\mathbf{wt}(u) \prec \mathbf{wt}(e_i(u)) = \mathbf{wt}(u) + \alpha_i$. $\square$

Proposition 2.5. If $\mathcal{B}$ is seminormal and $u \in \mathcal{B}$ is a highest weight element then $\mathbf{wt}(u) \in \Lambda^+$ is dominant.

Proof. If $\mathcal{B}$ is seminormal and $\mathbf{wt}(u)$ for $u \in \mathcal{B}$ is not dominant, then $\varphi_i(u) - \varepsilon_i(u) = \langle \mathbf{wt}(u), \alpha_i^\vee\rangle < 0$ for some $i \in I$, in which case $\varepsilon_i(u) > 0$ which means that $e_i(u) \neq 0$ so u is not a highest weight element. $\square$

In our category of normal crystals for the $\mathrm{GL}(n)$ Cartan type, all highest weights $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ were such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, and indeed all such elements are dominant weights.

Let $W = W(\Phi) \subset \mathrm{GL}(V)$ be the group generated by the reflections $s_i = r_{\alpha_i} : x \mapsto x - \langle x, \alpha_i^\vee\rangle\alpha_i$ for $i \in I$.

Proposition 2.6. Suppose $\mathcal{B}$ is a seminormal crystal. If $\mu, \nu \in \Lambda$ are weights in the same W-orbit, then there is a bijection between the sets $X := \{x \in \mathcal{B} : \mathbf{wt}(x) = \mu\}$ and $Y := \{x \in \mathcal{B} : \mathbf{wt}(x) = \nu\}$.

Proof. It suffices to prove this in the case when $\nu = s_i(\mu) = \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i$ for some fixed index $i \in I$.

The result is trivial if $\mu = \nu$ so assume $\mu \neq \nu$.

Then $k := \langle \mu, \alpha_i^\vee \rangle \neq 0$ and after possibly interchanging μ with ν we may assume that $k > 0$.

We claim the desired bijection $X \to Y$ is given by the operator f_i^k.

We have $f_i^k(x) \neq 0$ for all $x \in X$ since $\mathcal{B}$ is seminormal and $\varphi_i(x) = k + \varepsilon_i(x) \geq k$.

Thus, if $x \in X$ then the crystal axioms imply that $\mathbf{wt}(f_i^k(x)) = \mathbf{wt}(x) - k\alpha_i = \nu$, so $f_i^k(x) \in Y$.

The map $f_i^k : X \to Y$ is a bijection since it follows by similar arguments that $e_i^k : Y \to X$ is its inverse. $\square$

Example 2.7. If (Φ, Λ) is the $\mathrm{GL}(n)$ Cartan type, then

$$s_i(\lambda_1, \lambda_2, \dots, \lambda_n) = (\lambda_1, \lambda_2, \dots, \lambda_n) - (\lambda_i - \lambda_{i+1})(\mathbf{e}_i - \mathbf{e}_{i+1}) = (\lambda_1, \dots, \lambda_{i+1}, \lambda_i, \dots, \lambda_n).$$

The crystal $\mathrm{SSYT}_n(\lambda)$ has a unique element of highest weight $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and therefore also a unique element of weight $(\lambda_n, \dots, \lambda_2, \lambda_1)$. If $n = 5$ and $\lambda = (5, 3, 3, 1, 0)$ then these elements are

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 & 3 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 3 & 5 & 5 \\ \hline 3 & 4 & 4 \\ \cline{1-3} 4 & 5 & 5 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array}\ .$$

If you know about the representations of the Lie algebra $\mathfrak{sl}_2$, the following example will look familiar.

Example 2.8. Suppose (Φ, Λ) is the $\mathrm{GL}(2)$ Cartan type.

For each integer $k > 0$ there is a seminormal crystal $\mathcal{B}_{(k)}$ with $k + 1$ elements with crystal graph

$$v_k \xrightarrow{1} v_{k-2} \xrightarrow{1} v_{k-4} \xrightarrow{1} \cdots \xrightarrow{1} v_{-k+2} \xrightarrow{1} v_{-k}$$

The weight lattice is $\mathbb{Z}^2$ and we define $\mathbf{wt}(v_{k-2l}) = (k-l)\mathbf{e}_1 + l\mathbf{e}_2 = (k-l, l)$. Mapping v_{k-2l} to the 1-row semistandard tableau with k entries equal to 1 and l entries equal to 2 identifies $\mathcal{B}_{(k)}$ with $\mathrm{SSYT}_2((k))$.

Extending this example, there is also a finite type crystal $\mathcal{B}_\infty$ for Cartan type $\mathrm{GL}(2)$ that has infinitely many elements and is not seminormal. Its crystal graph is

$$v_0 \xrightarrow{1} v_{-2} \xrightarrow{1} v_{-4} \xrightarrow{1} \cdots$$

The weight map is $\mathbf{wt}(v_{-2k}) = -k\alpha_1 = (-k, k)$ and we set $\varphi_1(v_{-2k}) = -k$ and $\varepsilon_1(v_{-2k}) = k$.

3 Standard crystals for classical types

We have already seen the *standard crystal* $\mathbb{B}_n$ for type $\mathrm{GL}(n)$, which has crystal graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

and weight map $\mathbf{wt}(\boxed{i}) = \mathbf{e}_i$. We now describe the *standard crystals* for types B_n, C_n, and D_n.

For the root systems of types A_n, B_n, and D_n, there are two possibilities for the weight lattice Λ. However, one choice is always either a quotient or a subset of the other and it is usually not so important to pin down this choice explicitly.

Elements of the standard crystals will be boxed integers of the form $\boxed{i}$ for $i \geq 0$ or $\boxed{\bar{i}}$ for $i > 0$.

The notation $\bar{i}$ is just an abbreviation for $-i$ to make things more readable.

The element $\boxed{0} \in \mathcal{B}$ is distinct from the distinguished non-element $0 \notin \mathcal{B}$.

The weights of these elements will always be $\mathbf{wt}\left(\boxed{i}\right) = \mathbf{e}_i$ and $\mathbf{wt}\left(\boxed{\bar{i}}\right) = -\mathbf{e}_i$ and $\mathbf{wt}\left(\boxed{0}\right) = 0$.

The standard crystals will all be seminormal, so to define them we only need to give their crystal graphs.

Example 3.1 (Type B_n)**.** Recall that in type B_n the simple roots are

$$\alpha_i = \alpha_i^\vee = \mathbf{e}_i - \mathbf{e}_{i+1} \text{ for } i \in [n-1] \qquad \text{and} \qquad \alpha_n = \mathbf{e}_n \qquad \text{and} \qquad \alpha_n^\vee = 2\mathbf{e}_n.$$

The standard crystal in type B_n has crystal graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

Example 3.2 (Type C_n)**.** Recall that in type C_n the simple roots are

$$\alpha_i = \alpha_i^\vee = \mathbf{e}_i - \mathbf{e}_{i+1} \text{ for } i \in [n-1] \qquad \text{and} \qquad \alpha_n = 2\mathbf{e}_n \qquad \text{and} \qquad \alpha_n^\vee = \mathbf{e}_n.$$

The standard crystal in type C_n has crystal graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

Example 3.3 (Type D_n)**.** Recall that in type D_n the simple roots are

$$\alpha_i = \alpha_i^\vee = \mathbf{e}_i - \mathbf{e}_{i+1} \text{ for } i \in [n-1] \qquad \text{and} \qquad \alpha_n = \alpha_n^\vee = \mathbf{e}_{n-1} + \mathbf{e}_n.$$

The standard crystal in type D_n has crystal graph

$$\begin{tikzcd} & & & & \boxed{n} \arrow[dr, "n"] \\ \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \cdots \arrow[r, "n-2"] & \boxed{n-1} \arrow[ur, "n-1"] \arrow[dr, "n"'] & & \boxed{\overline{n-1}} \arrow[r, "n-2"] & \cdots \arrow[r, "2"] & \boxed{\bar{2}} \arrow[r, "1"] & \boxed{\bar{1}} \\ & & & & \boxed{\bar{n}} \arrow[ur, "n-1"'] \end{tikzcd}$$

Sometimes the following construction is useful for relating crystals or defining new crystals.

Definition 3.4. Let $\mathcal{B}$ be a crystal of finite type.

Let $\mathcal{B}^\vee = \{b^\vee : b \in \mathcal{B}\}$ be a set that is in bijection with $\mathcal{B}$. Define

$$\mathbf{wt}(b^\vee) = -\mathbf{wt}(b), \qquad e_i(b^\vee) = f_i(b)^\vee, \qquad f_i(b^\vee) = e_i(b)^\vee, \qquad \varepsilon_i(b^\vee) = \varphi_i(b), \qquad \varphi_i(b^\vee) = \varepsilon_i(b).$$

for all $i \in I$. These operators make $\mathcal{B}^\vee$ into a crystal, called the *dual* or *contragradient* of $\mathcal{B}$.

Example 3.5. If $\mathbb{B}_n$ is the standard crystal for $\mathrm{GL}(n)$ then the dual crystal $\mathbb{B}_n^\vee$ has crystal graph

$$\boxed{1^\vee} \xleftarrow{1} \boxed{2^\vee} \xleftarrow{2} \boxed{3^\vee} \xleftarrow{3} \cdots \xleftarrow{n-1} \boxed{n^\vee}$$

and weight map $\mathbf{wt}(\boxed{i^\vee}) = -\mathbf{e}_i$.

On the other hand, the standard crystals for types B_n, C_n, and D_n are all self-dual.

4 Tensor products and morphisms for general crystals

There is a natural tensor product for crystals of the same Cartan type.

Suppose $\mathcal{B}$ and $\mathcal{C}$ are crystals of Cartan type (Φ, Λ).

As a set, define $\mathcal{B} \otimes \mathcal{C} = \{b \otimes c : b \in \mathcal{B},\, c \in \mathcal{C}\}$. This is just a suggestive alternate notation for $\mathcal{B} \times \mathcal{C}$.

We extend the maps $\mathbf{wt}$ and e_i, f_i for $i \in I$ to $\mathcal{B} \otimes \mathcal{C}$ by essentially the same formulas as in type A:

- Define $\mathbf{wt}(b \otimes c) = \mathbf{wt}(b) + \mathbf{wt}(c) \in \Lambda$ for $b \in \mathcal{B}$ and $c \in \mathcal{C}$.
- Define $f_i(b \otimes c)$ to be either $f_i(b) \otimes c$ or $b \otimes f_i(c)$ according to the relative values of $\varphi_i(c)$ and $\varepsilon_i(b)$:

$$f_i(b \otimes c) := \begin{cases} f_i(b) \otimes c & \text{when } \varphi_i(c) \leq \varepsilon_i(b) \\ b \otimes f_i(c) & \text{when } \varphi_i(c) > \varepsilon_i(b). \end{cases}$$

- Define $e_i(b \otimes c)$ to be either $e_i(b) \otimes c$ or $b \otimes e_i(c)$ according to the relative values of $\varphi_i(c)$ and $\varepsilon_i(b)$:

$$e_i(b \otimes c) := \begin{cases} e_i(b) \otimes c & \text{when } \varphi_i(c) < \varepsilon_i(b) \\ b \otimes e_i(c) & \text{when } \varphi_i(c) \geq \varepsilon_i(b). \end{cases}$$

- In both of the preceding formulas we interpret $0 \otimes c = b \otimes 0 = 0 \notin \mathcal{B} \otimes \mathcal{C}$.

Unlike in our (seminormal) type A category, now we must also describe the string lengths for $\mathcal{B} \otimes \mathcal{C}$:

- Define $\varphi_i(b \otimes c)$ to be whichever of $\varphi_i(b)$ or $\varphi_i(c) + \langle \mathbf{wt}(b), \alpha_i^\vee \rangle$ is larger.
- Define $\varepsilon_i(b \otimes c)$ to be whichever of $\varepsilon_i(c)$ or $\varepsilon_i(b) - \langle \mathbf{wt}(c), \alpha_i^\vee \rangle$ is larger.
- If $\mathcal{B}$ and $\mathcal{C}$ are of finite type, meaning the string lengths never take value $-\infty$, then

$$\varphi_i(b \otimes c) = \varphi_i(b) + \max\Big\{0, \varphi_i(c) - \varepsilon_i(b)\Big\} \qquad \text{and} \qquad \varepsilon_i(b \otimes c) = \varepsilon_i(c) + \max\Big\{0, \varepsilon_i(b) - \varphi_i(c)\Big\}.$$

Going along with these definitions are two very desirable, but somewhat technical, propositions:

Proposition 4.1. Relative to the maps $\mathbf{wt}$, e_i, f_i, ε_i, φ_i just defined, $\mathcal{B} \otimes \mathcal{C}$ is a crystal.

If $\mathcal{B}$ and $\mathcal{C}$ are seminormal normals for Cartan type (Φ, Λ), then $\mathcal{B} \otimes \mathcal{C}$ is also a seminormal crystal.

An *isomorphism* between crystals $\mathcal{B}$ and $\mathcal{C}$ for Cartan type (Φ, Λ) is a weight-preserving bijection $\mathcal{B} \to \mathcal{C}$ that commutes with all crystal operators and string lengths in the way you would expect. Such a map corresponds to an isomorphism of labeled directed graphs between the corresponding crystal graphs.

Proposition 4.2. Suppose $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are crystals for Cartan type (Φ, Λ).

The bijection $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D} \to \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$ mapping $(b \otimes c) \otimes d \mapsto b \otimes (c \otimes d)$ is a crystal isomorphism.

The proofs of both propositions can be found in Chapter 2 of Bump and Schilling's book. The details that need to be checked are straightforward calculations.

It is also possible to write down relatively explicit formulas for the crystal operators and string lengths of k-fold tensor products of crystals $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \cdots \otimes \mathcal{B}_k$. This leads quickly to the signature rule for the crystals operators of $\mathbb{B}_n^{\otimes m}$. This is also done in Bump and Schilling's book but we will skip the details.

We should also define a general *morphism* between crystals $\mathcal{B}$ and $\mathcal{C}$ of the same Cartan type.

A map $\psi : \mathcal{B} \to \mathcal{C} \sqcup \{0\}$ is a *morphism* if the following properties hold for all $i \in I$ and $b \in \mathcal{B}$:

- We have $\mathbf{wt}(\psi(b)) = \mathbf{wt}(b)$ and $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$ and $\varphi_i(\psi(b)) = \varphi_i(b)$ whenever $\psi(b) \neq 0$.
- We have $e_i(\psi(b)) = \psi(e_i(b))$ whenever $\psi(b) \neq 0$ and $e_i(b) \neq 0$ and $\psi(e_i(b)) \neq 0$.
- We have $f_i(\psi(b)) = \psi(f_i(b))$ whenever $\psi(b) \neq 0$ and $f_i(b) \neq 0$ and $\psi(f_i(b)) \neq 0$.

In words: ψ preserves weights and string lengths and commutes with crystal operators in all cases except when this doesn't make sense (because something is mapped to zero).

With respect to this notion of a morphism, crystals for a given Cartan type form a category.

(This means that the identity map $\mathcal{B} \to \mathcal{B}$ is a morphism and compositions of morphisms are morphisms.)

A morphism $\psi : \mathcal{B} \to \mathcal{C}$ is *strict* if it always commutes with the crystal operators, in the sense that

$$e_i(\psi(b)) = \psi(e_i(b)) \qquad \text{and} \qquad f_i(\psi(b)) = \psi(f_i(b))$$

for all $i \in I$ and $b \in \mathcal{B}$, interpreting $\psi(0) = e_i(0) = f_i(0) = 0$.

Example 4.3. Suppose $\mathcal{B}$ is a crystal for Cartan type (Φ, Λ) with weight map $\mathbf{wt}$.

Suppose $\lambda \in \Lambda$ is a weight that has $\langle \lambda, \alpha_i^\vee \rangle = 0$ for all $i \in I$.

For example, in Cartan type $\mathrm{GL}(n)$, we could take λ to be any integer multiple of $\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n$.

Replacing the weight map of $\mathcal{B}$ by the new function $\mathbf{wt}'(b) := \mathbf{wt}(b) + \lambda$, but retaining the same crystal operators and string lengths, gives another crystal structure for Cartan type (Φ, Λ).

We call this new crystal the *twist* of $\mathcal{B}$ by λ; it is isomorphic to $\mathcal{T}_\lambda \otimes \mathcal{B}$ with $\mathcal{T}_\lambda$ as in Example 2.2.

Example 4.4. Here is an example of a crystal morphism that is not strict in Cartan type $\mathrm{GL}(2)$.

Recall $\mathcal{B}_{(k)}$ and $\mathcal{B}_\infty$ from Example 2.8. Define $\psi : \mathcal{B}_{(k)} \to \mathcal{T}_{(k,0)} \otimes \mathcal{B}_\infty$ by $\psi(v_{k-2l}) = t_{(k,0)} \otimes v_{-2l}$.

One can check that this map is a morphism, but $f_1(\psi(v_{-k})) = t_{(k,0)} \otimes v_{-2k-2} \neq 0 = \psi(0) = \psi(f_1(v_{-k}))$.

5 Root strings and characters of general crystals

Suppose $\mathcal{B}$ is a seminormal crystal for Cartan type (Φ, Λ). Fix an index $i \in I$.

Consider the equivalence relation on $\mathcal{B}$ that has $x \sim y$ if $x = e_i^k(y)$ or $x = f_i^k(y)$ for some $k \geq 0$.

An equivalence class under this relation is called an *i-root string*.

Each i-root string is finite (as $\mathcal{B}$ is seminormal) and its elements may be listed as $u_0, u_1, \dots, u_m$ where

$$e_i(u_0) = 0, \qquad u_k = f_i^k(u_0), \qquad \mathbf{wt}(u_k) = \mathbf{wt}(u_0) + k\alpha_i, \qquad \text{and} \qquad m = \varphi_i(u_0).$$

Recall that $s_i \in \mathrm{GL}(V)$ for $i \in I$ denotes the reflection that acts on Λ by $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$.

Given $b \in \mathcal{B}$ with $k = \langle \mathbf{wt}(b), \alpha_i^\vee \rangle$, define $\sigma_i(b) = \begin{cases} f_i^k(b) & \text{if } k \geq 0 \\ e_i^{-k}(b) & \text{if } k < 0. \end{cases}$

Proposition 5.1. The map σ_i reverses each i-root string in $\mathcal{B}$ and $\mathbf{wt}(\sigma_i(b)) = s_i(\mathbf{wt}(b))$ for each $b \in \mathcal{B}$.

Proof. This generalizes a result for type A crystals from Lecture 2. The proof is essentially the same. □

Let $\mathcal{E}$ be the free abelian group spanned by the symbols t^λ for $\lambda \in \Lambda$.

The Weyl group $W = \langle s_i : i \in I \rangle$ acts linearly on $\mathcal{E}$ by the formula $s_i \cdot t^\lambda = t^{s_i(\lambda)}$.

If $\mathcal{B}$ has finitely many elements, then its *character* is $\mathrm{ch}(\mathcal{B}) = \sum_{b \in \mathcal{B}} t^{\mathbf{wt}(b)} \in \mathcal{E}$.

This recovers our notion of characters in Cartan type $\mathrm{GL}(n)$ since then we can identify $\mathcal{E} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Corollary 5.2. If $\mathcal{B}$ is finite then its character is invariant under the action of W on $\mathcal{E}$.

Proof. The previous proposition implies that $s_i(\mathrm{ch}(\mathcal{B})) = \mathrm{ch}(\mathcal{B})$ since each σ_i is a bijection $\mathcal{B} \to \mathcal{B}$. □