1 Last time: crystals for general Cartan types

Fix a root system $\Phi \subset V$ with simple roots $\{\alpha_i : i \in I\}$ and weight lattice Λ. Write $\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$.

A *crystal* of type (Φ, Λ) is a nonempty set $\mathcal{B}$ with maps

$$\mathbf{wt} : \mathcal{B} \to \Lambda, \qquad e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \text{ for } i \in I, \qquad \text{and} \qquad \varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\} \text{ for } i \in I,$$

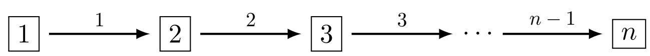
where $0 \notin \mathcal{B}$ is an auxiliary element, satisfying two main axioms.

A crystal comes with notions of a *crystal graph*, *full subcrystals*, and *highest weights*. A crystal is *seminormal* if φ_i & ε_i measure how many times you can apply f_i & e_i to an element before reaching zero.

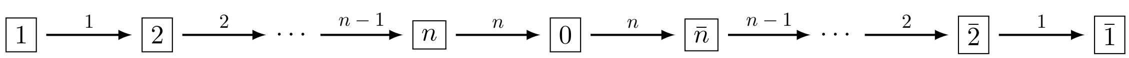
Possibly useful fact: if Φ spans the ambient space V and the string lengths never take value $-\infty$ (i.e., Φ is *semisimple* and the crystal is of *finite type*), then the crystal graph of $\mathcal{B}$ determines $\mathbf{wt}$.

Example 1.1. Last time we introduced *standard crystals* for all classical Cartan types:

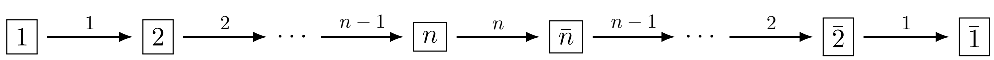
- Type A_{n-1}:



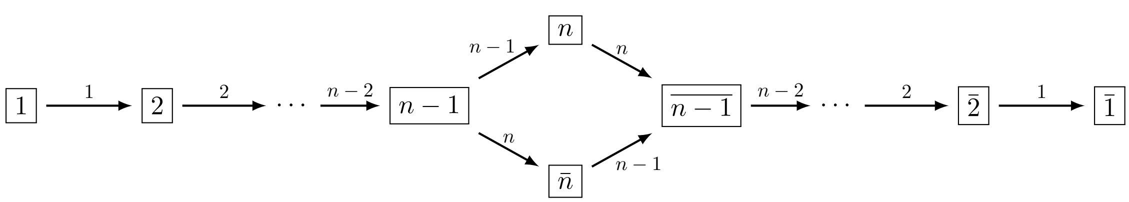
- Type B_n:



- Type C_n:



- Type D_n:



Each of these crystals is seminormal (so the string lengths are determined by the crystal graph), and the weight map is defined by $\mathbf{wt}(\boxed{i}) = \mathbf{e}_i$ and $\mathbf{wt}(\boxed{\bar{i}}) = -\mathbf{e}_i$ and $\mathbf{wt}(\boxed{0}) = 0$.

A *morphism* between crystals is a map $\mathcal{B} \to \mathcal{C} \sqcup \{0\}$ that preserves weights and string lengths and commutes with the crystal operators in all cases when the relevant conditions make sense. A morphism is an isomorphism if it induces a bijection between crystal graphs. With this notion of morphisms, crystals for a given Cartan type form a monoidal category with a tensor product $\otimes$.

The formulas for the weight map, crystal operators, and string lengths of a tensor product of crystals $\mathcal{B} \otimes \mathcal{C}$ are similar to what we saw in the first few lectures for the (seminormal) type A case.

Finally, we discussed crystal *characters*. Let $\mathcal{E}$ be the free abelian group spanned by t^λ for $\lambda \in \Lambda$.

Let $s_i \in \mathrm{GL}(V)$ be the reflection $x \mapsto x - \langle x, \alpha_i^\vee \rangle \alpha_i$. Then $s_i(\Lambda) = \Lambda$.

The Weyl group $W := \langle s_i : i \in I \rangle$ acts linearly on $\mathcal{E}$ by the formula $s_i \cdot t^\lambda = t^{s_i(\lambda)}$.

If $\mathcal{B}$ is finite then its *character* is $\mathrm{ch}(\mathcal{B}) = \sum_{b \in \mathcal{B}} t^{\mathbf{wt}(b)}$. This is aways a W-invariant element of $\mathcal{E}$.

2 Product Cartan types and Levi branching

Let Φ_1 and Φ_2 be root systems in ambient spaces V_1 and V_2 with weight lattices Λ_1 and Λ_2.

Define $V = V_1 \oplus V_2$ to be the orthogonal direct sum of the real inner product spaces V_i.

Then let $\Lambda = \Lambda_1 \oplus \Lambda_2$ and $\Phi = \Phi_1 \sqcup \Phi_2$.

(One sometimes writes $\Phi_1 \oplus \Phi_2$ in place of the disjoint union $\Phi_1 \sqcup \Phi_2$.)

The pair (Φ, Λ) then consists of a root system $\Phi \subset V$ with a weight lattice Λ, so is a Cartan type.

If X and Y are the Cartan types of (Φ_1, Λ_1) and (Φ_2, Λ_2), then we say that (Φ, Λ) has type $X \times Y$.

As a special case, if $X = Y = \mathrm{GL}(n)$ then we say that (Φ, Λ) has type $\mathrm{GL}(n) \times \mathrm{GL}(N)$ or $A_{n-1} \times A_{n-1}$.

Suppose $\mathcal{B}_1$ is a crystal for Cartan type (Φ_1, Λ_1) and $\mathcal{B}_2$ is a crystal for Cartan type (Φ_2, Λ_2)

We can form a crystal $\mathcal{B}_1 \times \mathcal{B}_2$ for Cartan type $(\Phi, \Lambda) = (\Phi_1 \sqcup \Phi_2, \Lambda_1 \oplus \Lambda_2)$.

The elements of $\mathcal{B}_1 \times \mathcal{B}_2$ are pairs (b_1, b_2) with $b_i \in \mathcal{B}_i$.

The weight map is $\mathbf{wt}(b_1, b_2) = \mathbf{wt}(b_1) + \mathbf{wt}(b_2)$.

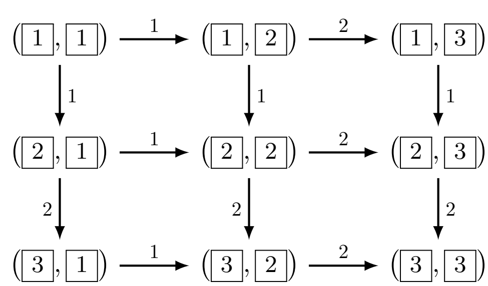
If i indexes a simple root $\alpha_i \in \Phi_1$, then $f_i(b_1, b_2) = (f_i(b_1), b_2)$ and $\varphi_i(b_1, b_2) = \varphi_i(b_1)$

If i indexes a simple root $\alpha_i \in \Phi_2$, then $f_i(b_1, b_2) = (b_1, f_i(b_2))$ and $\varphi_i(b_1, b_2) = \varphi_i(b_2)$.

Here we interpret $(0, b_2) = (b_1, 0) = 0$. The maps e_i and ε_i are defined analogously.

Note that $\mathcal{B}_1 \times \mathcal{B}_2$ is not the same as $\mathcal{B}_1 \otimes \mathcal{B}_2$, although the crystals are in bijections as sets.

Example 2.1. The crystal graph of $\mathbb{B}_3 \times \mathbb{B}_3$, where $\mathbb{B}_3$ is the standard crystal of type A_2, is



This crystal is of type $A_2 \times A_2$, whereas $\mathbb{B}_3 \otimes \mathbb{B}_3$ is still of type A_2.

By erasing some arrows from the crystal graph, we can view a crystal for a given Cartan type (Φ, Λ) as a crystal for a "smaller" Cartan type (Φ', Λ), with root system $\Phi' \subset \Phi$.

This process, called *branching*, may turn a connected crystal of type (Φ, Λ) into a disconnected crystal of type (Φ', Λ). Crystals correspond to representations of Lie groups and this kind of branching reflects what happens when a representation is restricted to a Levi subgroup.

We can describe the branching process more explicitly.

Suppose (Φ, Λ) is a Cartan type with corresponding simple roots $\{\alpha_i : i \in I\}$.

Let $J \subset I$ be any subset of the indexing set for the simple roots (and fundamental weights).

The corresponding *Levi root sytem* is then $\Phi_J := \Phi \cap \mathbb{Z}\text{-span}\{\alpha_i : i \in J\}$.

Recall that $s_i \in \mathrm{GL}(V)$ denotes the reflection $x \mapsto x - 2\frac{\langle x, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i$ and $W = \langle s_i : i \in I \rangle$.

Define $W_J = \langle s_i : i \in J \rangle \subset W$. One can show that $\Phi_J = \{\alpha \in \Phi : \alpha = w(\alpha_i) \text{ for some } w \in W_J,\ i \in J\}$.

Example 2.2. Suppose Φ is type A_3 root system with 12 elements

$$\Phi = \{\pm(\mathbf{e}_1 - \mathbf{e}_2), \pm(\mathbf{e}_1 - \mathbf{e}_3), \pm(\mathbf{e}_1 - \mathbf{e}_4), \pm(\mathbf{e}_2 - \mathbf{e}_3), \pm(\mathbf{e}_2 - \mathbf{e}_4), \pm(\mathbf{e}_3 - \mathbf{e}_4)\}$$

and simple roots $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i \in I = \{1, 2, 3\}$. If $J = \{1, 3\}$ then

$$\Phi_J = \{\pm(\mathbf{e}_1 - \mathbf{e}_2)\} \sqcup \{\pm(\mathbf{e}_3 - \mathbf{e}_4)\}$$

so Φ_J has type $A_1 \times A_1$.

Example 2.3. It is a standard exercise to observe that if Φ has classical type B_n, C_n, or D_n, so that our index set is $I = [n]$, then Φ_J for $J = [n-1]$ is just the root system of type A_{n-1}.

If you know about *Dynkin diagrams*, then observe that the Levi root system Φ_J may be read off from the Dynkin diagram of Φ by erasing the vertices indexed by $i \notin J$ along with any incident edges. See §2.8 of Bump and Schilling's book for more details.

To form a Cartan type from the Levi root system, one can take the same ambient space V and weight lattice Λ. In particular, (Φ_J, Λ) is another Cartan type.

Suppose $\mathcal{B}$ is a crystal for Cartan type (Φ, Λ). To obtain a crystal for Cartan type (Φ_J, Λ), we simply forget the maps f_i, e_i, φ_i, and ε_i for all indices $i \notin J$. The resulting crystal structure is called the *(Levi) branched crystal* and is sometimes denoted $\mathcal{B}_J$. The weight map of the new crystal is unchanged, and its crystal graph is obtained from the graph of $\mathcal{B}$ by erasing all edges labeled by $i \notin J$.

Example 2.4. Let Φ be the classical root system of type B_n, C_n, or D_n so $I = [n]$.

Suppose $J = [n-1]$ and $\mathcal{B}$ is the standard crystal for Φ.

As usual write $\mathbb{B}_n$ for the standard crystal of type A_{n-1}, i.e., Cartan type $\mathrm{GL}(n)$.

Then $\mathcal{B}_J$ is isomorphic to $\mathbb{B}_n \sqcup \mathbb{B}_n^\vee$ in types B_n and D_n, and to $\mathbb{B}_n \sqcup \mathbb{B}_n^\vee \sqcup \mathcal{T}_0$ in type C_n.

3 Normal crystals

A root system Φ is *simply-laced* if all roots $\alpha \in \Phi$ have the same length.

This occurs in classical types A_n, D_n and exceptional types E_6, E_7, E_8, but *not* in types B_n, C_n.

Let $\mathcal{B}$ be a crystal for Cartan type $\mathrm{GL}(n)$.

Recall our notion of *normal crystals* from a few lectures ago:

- $\mathcal{B}$ is *normal* if each of its full subcrystals is isomorphic to a full subcrystal of $\mathbb{B}_n^{\otimes m}$ for some $m \geq 1$.
- In fact, every connected normal crystal $\mathcal{B}$ is isomorphic to $\mathrm{SSYT}_n(\lambda)$ for some partition λ.
- Consequently, each connected normal crystal $\mathcal{B}$ has a unique highest weight element.

  The highest weight of this element is always the (unique) partition $\lambda \in \mathbb{N}^n$ such that $\mathcal{B} \cong \mathrm{SSYT}_n(\lambda)$.

Our main topic today is a completely different way of characterizing normal crystals.

Specifically, we will see that a normal crystal can be detected by examining its branched crystals just to Levi root systems of rank two. An advantage of this approach, pioneered by Stembridge around 2003, is that it will generalize immediately to all simply-laced root systems.

We have some freedom in how we define "normal" crystals of an arbitrary Cartan type.

What do we want from such crystals?

First let $\mathcal{B}$ be a crystal for an arbitrary Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$.

Recall that a *highest weight element* $u \in \mathcal{B}$ is an element with $e_i(u) = 0$ for all $i \in I$.

Define a partial order $\preceq$ on $\mathcal{B}$ by setting $y \preceq x$ if a path connects x to y in the crystal graph.

This means that $y \preceq x$ if and only if $x = e_{i_1} e_{i_2} \cdots e_{i_k}(y)$ for some indices $i_1, i_2, \ldots i_k \in I$, in which case

$$\mathbf{wt}(x) = \mathbf{wt}(y) + \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k}.$$

Thus $\mathbf{wt}(y) \preceq \mathbf{wt}(x)$ where $\preceq$ is the partial order Λ with $\lambda \prec \lambda + \alpha_i$ for all $i \in I$.

(We may have $\mathbf{wt}(y) \preceq \mathbf{wt}(x)$ without $y \preceq x$, however.)

Observation 3.1. If $\mathcal{B}$ has a unique highest weight element u then $x \preceq u$ for all $x \in \mathcal{B}$ so $\mathcal{B}$ is connnected.

Proof. Any $u \in \mathcal{B}$ that is maximal under $\preceq$ is clearly a highest weight element. □

For crystals of Cartan type (Φ, Λ), "normal" crystals should solve the following problem:

> Associate to each dominant weight $\lambda \in \Lambda^+$ a (necessarily connected) seminormal crystal $\mathcal{B}_\lambda$ with a unique highest weight element u_λ satisfying $\mathbf{wt}(u_\lambda) = \lambda$. Construct this correspondence in such a way that each connected component of $\mathcal{B}_\lambda \otimes \mathcal{B}_\mu$ for $\lambda, \mu \in \Lambda^+$ is isomorphic to $\mathcal{B}_\nu$ for some $\nu \in \Lambda^+$.

The category of all seminormal crystals is too large to satisfy these requirements.

4 Stembridge crystals

Fix a Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$. **We assume Φ is simply-laced.**

In this section we will define a full subcategory of crystals for type (Φ, Λ) in terms of some axioms the depend solely on the the structure of branched crystals to Levi root systems of rank two.

Fix distinct indices $i, j \in I$. Because Φ is simply-laced, we have either $\langle \alpha_i, \alpha_j \rangle = 0$ or $\langle \alpha_i, \alpha_j \rangle = -1$.

Choose a crystal $\mathcal{B}$ of type (Φ, Λ) and let $J = \{i, j\}$. The branched crystal $\mathcal{B}_J$ has one of two forms.

If α_i and α_j are orthogonal then $\mathcal{B}_J$ has type $A_1 \times A_1$.

Every type A_1 crystal graph is a disjoint union of paths $\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$.

Thus a type $A_1 \times A_1$ crystal graph is a disjoint union of rectangles given by Cartesian products of paths.

Remark. This is consistent with the fact that all connected (normal) crystals of type A_1 (type GL(2)) are isomorphic to crystals of semistandard tableaux $\mathrm{SSYT}_2(\lambda)$. The partition λ is required to have at most 2 parts. If $\lambda = (k)$ has only one part then the crystal graph of $\mathrm{SSYT}_2(\lambda)$ is clearly a path:

$$\boxed{1\,|\,1\,|\,1} \xrightarrow{1} \boxed{1\,|\,1\,|\,2} \xrightarrow{1} \boxed{1\,|\,2\,|\,2} \xrightarrow{1} \boxed{2\,|\,2\,|\,2}.$$

In general, if $\lambda \in \mathbb{N}^n$ is a partition and $\eta = (1, 1, \ldots, 1) \in \mathbb{N}^n$, then $\lambda + \eta$ is a partition and

$$\mathrm{SSYT}_n(\lambda + \eta) \cong \mathrm{SSYT}_n(\lambda) \otimes \mathcal{T}_\eta.$$

This is because if we add column with n boxes to a semistandard tableau $T \in \mathrm{SSYT}_n(\lambda)$ then the entries in these boxes must be $1, 2, 3, \dots, n$ consecutively:

$$\ydiagram{4,3,2} \in \mathrm{SSYT}_3(\lambda) \quad \rightsquigarrow \quad \begin{ytableau} 1 & \ & \ & \ & \ \\ 2 & \ & \ & \ \\ 3 & \ & \ \end{ytableau} \in \mathrm{SSYT}_3(\lambda + \eta).$$

Thus if $\lambda = (j + k, j)$ then $\mathrm{SSYT}_2(\lambda) \cong \mathrm{SSYT}_n((k)) \otimes \mathcal{T}_{(j,j)}$ still has crystal graph which is a path.

If α_i and α_j are not orthogonal then $\mathcal{B}_J$ is a crystal of type A_2 (type GL(3)).

More can happen in this case; e.g., $\mathcal{B}_J$ could be isomorphic to $\mathrm{SSYT}_3(\lambda)$ for any partition $\lambda \in \mathbb{N}^3$.

The axioms governing our category of interest are somewhat technical. I will not spend as much time motivating these axioms as Bump and Schilling do in Chapter 4 of their book. I am also going to omit most of the relevant proofs of the important properties of the result crystals. These arguments can get somewhat complicated, though everything ultimately boils down to accessible computations.

Definition 4.1. A crystal $\mathcal{B}$ of simply-laced Cartan type (Φ, Λ), with simple roots $\{\alpha_i : i \in I\}$, is a *weak Stembridge crystal* if the following axioms hold for all distinct indices $i, j \in I$:

(S0) If $e_i(x) = 0$ then $\varepsilon_i(x) = 0$.

(S1) Suppose $x, y \in \mathcal{B}$ and $y = e_i(x)$. Then $\varepsilon_j(y) - \varepsilon_j(x) \in \{0, 1\}$.

In the case when α_i and α_j are orthogonal, $\varepsilon_j(y) = \varepsilon_j(x)$.

(S2) If $x \in \mathcal{B}$ with $\varepsilon_i(x) > 0$ and $\varepsilon_j(e_i(x)) = \varepsilon_j(x) > 0$ then $e_i e_j(x) = e_j e_i(x)$ and $\varphi_i(e_j(x)) = \varphi_i(x)$.

(S3) Suppose $x \in \mathcal{B}$ has $\varepsilon_j(e_i(x)) = \varepsilon_j(x) + 1 > 1$ and $\varepsilon_i(e_j(x)) = \varepsilon_i(x) + 1 > 1$. Then

$$e_j e_i^2 e_j(x) = e_i e_j^2 e_i(x) \neq 0, \qquad \varphi_i(e_j(x)) = \varphi_i(e_j^2 e_i(x)), \qquad \text{and} \qquad \varphi_j(e_i(x)) = \varphi_j(e_i^2 e_j(x)).$$

We also require the following dual axioms:

(S0′) If $f_i(x) = 0$ then $\varphi_i(x) = 0$.

(S1′) Suppose $x, y \in \mathcal{B}$ and $y = f_i(x)$. Then $\varphi_j(y) - \varphi_j(x) \in \{0, 1\}$.

In the case when α_i and α_j are orthogonal, $\varphi_j(y) = \varphi_j(x)$.

(S2′) If $x \in \mathcal{B}$ with $\varphi_i(x) > 0$ and $\varphi_j(f_i(x)) = \varphi_j(x) > 0$ then $f_i f_j(x) = f_j f_i(x)$ and $\varepsilon_i(f_j(x)) = \varepsilon_i(x)$.

(S3′) Suppose $x \in \mathcal{B}$ has $\varphi_j(f_i(x)) = \varphi_j(x) + 1 > 1$ and $\varphi_i(f_j(x)) = \varphi_i(x) + 1 > 1$. Then

$$f_j f_i^2 f_j(x) = f_i f_j^2 f_i(x) \neq 0, \qquad \varepsilon_i(f_j(x)) = \varepsilon_i(f_j^2 f_i(x)), \qquad \text{and} \qquad \varepsilon_j(f_i(x)) = \varepsilon_j(f_i^2 f_j(x)).$$

If $\mathcal{B}$ is seminormal then axioms (S0) and (S0′) are automatically satisfied.

A *Stembridge crystal* is a weak Stembridge crystal $\mathcal{B}$ that is also seminormal.

The existence of these simple (but not too simple) local axioms is conceptually appealing, and sometimes checking the axioms is an effective proof strategy.

Theorem 4.2. If $\mathcal{B}$ and $\mathcal{C}$ are Stembridge crystals for the same Cartan type then so is $\mathcal{B} \otimes \mathcal{C}$.

Checking this result directly from the Stembridge axioms is one of the more complicated proofs in Bump and Schilling's book. In Stembridge's original paper, the axioms were given to characterize a family of crystals that was already known to be closed under tensor products, using connections representation theory that we have so far not discussed.

Morphisms between Stembridge crystals are crystal morphisms in the usual sense.

It follows that Stembridge crystals form a *monoidal category*.

Theorem 4.3. Let $\mathbb{B}_n$ be the standard crystal of type A_{n-1} or D_n. Then $\mathbb{B}_n$ is a Stembridge crystal.

Consequently, any full subcrystal of $\mathbb{B}_n^{\otimes m}$ for some m is a Stembridge crystal.

Proof. Checking that $\mathbb{B}_n$ satisfies the Stembridge axioms is an easy exercise.

Tensor powers of a Stembridge crystal are Stembridge crystals by the previous theorem.

By construction, if a crystal satisfies the Stembridge axioms then so do all of its full subcrystals. □

The following results are fundamental. The proofs from the Stembridge are not as complicated as checking that tensor products of Stembridge crystals are Stembridge crystals; see Bump and Schilling's book.

Theorem 4.4. A connected Stembridge crystal has a unique highest weight element.

The *highest weight* of a crystal with highest weight element u is $\mathbf{wt}(u) \in \Lambda^+$.

Theorem 4.5. If two connected Stembridge crystals for the same Cartan type have the same unique highest weights, then the crystals are isomorphic.

Recall that if η is in the orthogonal complement of $\mathbb{R}\Phi$ in V then we can twist a crystal by adding η to the weights of all elements.

Corollary 4.6. In type A_{n-1}, all normal crystals are Stembridge crystals and all Stembridge crystals are twists of normal crystals. In other words, up to twisting, the categories of normal crystals and Stembridge crystals for Cartan type $\mathrm{GL}(n)$ are the same.

Proof. Every normal crystal is a Stembridge crystal by Theorem 4.3.

A connected Stembridge crystal $\mathcal{B}$ in type A_{n-1} has a unique highest weight λ.

This weight is dominant but might fail to be a partition if $\lambda_n < 0$.

However, we can certainly twist $\mathcal{B}$ by a weight so that the highest weight becomes a partition

The twisted crystal is then normal since if its highest weight is λ then it must be isomorphic to the Stembridge crystal $\mathrm{SSYT}_n(\lambda)$. □

This leads to the following simpler, but less explicit, way of phrasing the Stembridge axioms:

Corollary 4.7. A crystal $\mathcal{B}$ of simply-laced Cartan type (Φ, Λ), with simple roots $\{\alpha_i : i \in I\}$, is a *Stembridge crystal* if and only if $\mathcal{B}$ is seminormal and for all distinct indices $i, j \in I$:

- If $\langle \alpha_i, \alpha_j \rangle = 0$ then each full subcrystal of $\mathcal{B}_J$ for $J = \{i, j\}$ has crystal graph that is a rectangle.
- If $\langle \alpha_i, \alpha_j \rangle \neq 0$ then each full subcrystal of $\mathcal{B}_J$ for $J = \{i, j\}$ has crystal graph $\cong$ to some $\mathrm{SSYT}_3(\lambda)$.