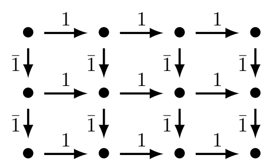
1 Last time: Stembridge crystals

Remark: whenever we say "(Cartan) type A_{n-1}" we mean the same thing as "(Cartan) type $\mathrm{GL}(n)$."

Suppose (Φ_1, Λ_1) and (Φ_2, Λ_2) are Cartan types. Then $(\Phi, \Lambda) = (\Phi_1 \sqcup \Phi_2, \Lambda_1 \oplus \Lambda_2)$ is also a Cartan type. If $\mathcal{B}_1$ is a crystal for Cartan type (Φ_1, Λ_1) and $\mathcal{B}_2$ is a crystal for Cartan type (Φ_2, Λ_2) then we can form a crystal $\mathcal{B}_1 \times \mathcal{B}_2$ for Cartan type (Φ, Λ). The elements of $\mathcal{B}_1 \times \mathcal{B}_2$ are pairs (b_1, b_2) with $b_i \in \mathcal{B}_i$.

Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are connected seminormal crystals of type A_1. The crystal graphs of these crystals are paths of the form $\bullet \xrightarrow{1} \bullet \xrightarrow{1} \bullet \cdots \bullet \xrightarrow{1} \bullet$. Let 1 and $\bar{1}$ be the indices for the distinct simple roots in the root system of type $A_1 \times A_1$. Then $\mathcal{B}_1 \times \mathcal{B}_2$ has a crystal graph that looks like



(This picture corresponds to $|\mathcal{B}_1| = 4$ and $|\mathcal{B}_2| = 3$.) We say that a type $A_1 \times A_1$ crystal is a *rectangle* if its crystal graph is isomorphic to a Cartesian product of this form. We ignore the weight map in this definition. Not every $A_1 \times A_1$ crystal is a rectangle: one could have a crystal graph like $\bullet \xleftarrow{\bar{1}} \bullet \xrightarrow{1} \bullet$.

Suppose (Φ, Λ) is a Cartan type with corresponding simple roots $\{\alpha_i : i \in I\}$. Let $J \subset I$. Then $\Phi_J := \Phi \cap \mathbb{Z}\text{-span}\{\alpha_i : i \in J\}$ is a root system and (Φ_J, Λ) is another Cartan type. Suppose $\mathcal{B}$ is a crystal for Cartan type (Φ, Λ). To obtain a crystal for Cartan type (Φ_J, Λ), we simply forget the maps f_i, e_i, φ_i, and ε_i for all indices $i \notin J$. We denote resulting crystal structure by $\mathcal{B}_J$.

It is convenient to modify our definition of a *normal crystal* as follows:

Definition 1.1. A type A_{n-1} crystal is *normal* if each of its full subcrystals is isomorphic to a **twist** of a full subcrystal of $\mathbb{B}_n^{\otimes m}$ for some m. (The twisting part of this definition is different from last time.)

Recall that twisting just refers to altering the weight map by adding some $\eta \in \mathbb{Z}\text{-span}\{\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n\}$ to all values. The crystal obtained by twisting $\mathcal{B}$ by η is isomorphic to $\mathcal{B} \otimes \mathcal{T}_\eta$ where $\mathcal{T}_\eta$ is the 1-element crystal with unique weight η.

Theorem 1.2. Each connected, normal type A_{n-1} crystal is isomorphic to $\mathrm{SSYT}_n(\lambda) \otimes \mathcal{T}_\eta$ for a unique choice of a partition λ with **at most** $n-1$ **parts** and a weight $\eta \in \mathbb{Z}\text{-span}\{\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n\}$.

Proof. We know that such a crystal is isomorphic to $\mathrm{SSYT}_n(\lambda) \otimes \mathcal{T}_\eta$ for some choice of λ and η, and that it has a unique highest weight element. If the weight of this element is $w = (w_1, w_2, \dots, w_n) \in \mathbb{Z}^n$, then w is dominant and we can take $\eta = w_n \cdot (\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n)$ and $\lambda = w - \eta$. □

Remark. Recall that the dual $\mathcal{B}^\vee$ of a crystal $\mathcal{B}$ is obtained interchanging all crystal operators $e_i \leftrightarrow f_i$ and string lengths $\varepsilon_i \leftrightarrow \varphi_i$ and negating the weight map. This reverses all arrows in the crystal graph. The dual $\mathbb{B}_n^\vee$ of the standard crystal of type A_{n-1} is connected but not isomorphic to any $\mathrm{SSYT}_n(\lambda)$ because its weights all have negative coefficients. However, we do have $\mathbb{B}_n^\vee \cong \mathrm{SSYT}_n(\lambda) \otimes \mathcal{T}_\eta$ for some λ and η, so $\mathbb{B}_n^\vee$ is a normal crystal under our new definition. This will be explored further in HW2.

Now suppose (Φ, Λ) is a *simply-laced* Cartan type with simple roots $\{\alpha_i : i \in I\}$. This means that all roots in Φ have the same length, so if Φ is irreducible then it has type A, D, or E.

Definition 1.3. A crystal $\mathcal{B}$ of simply-laced Cartan type (Φ, Λ) is a *Stembridge crystal* if $\mathcal{B}$ is seminormal and for each 2-element subset $J = \{i, j\} \subseteq I$, the following holds:

(1) If $\langle \alpha_i, \alpha_j \rangle = 0$ then each full subcrystal of $\mathcal{B}_J$ is a type $A_1 \times A_1$ crystal that is a rectangle.

(2) If $\langle \alpha_i, \alpha_j \rangle \neq 0$ then each full subcrystal of $\mathcal{B}_J$ is a type A_2 crystal that is normal.

This was our second characterization of Stembridge crystals from last lecture. Our first definition consisted of eight *Stembridge axioms* stated in terms of the operators $e_i, e_j, f_i, f_j, \varepsilon_i, \varepsilon_j, \varphi_i, \varphi_j$. Key facts:

- The standard crystals for types A_{n-1} and D_n are Stembridge crystals.
- Tensor products of Stembridge crystals are Stembridge crystals.
- Each connected Stembridge crystal has a unique highest weight element.
- Two connected Stembridge crystals of the same type with the same highest weight are isomorphic.
- In type A_{n-1}, Stembridge crystals are the same thing as normal crystals (as defined today).

Lemma 1.4. Suppose $\mathcal{B}$ is a Stembridge crystal of type (Φ, Λ) and $i, j \in I$ are such that $\langle \alpha_i, \alpha_j \rangle = 0$. Then the crystal operators e_i and e_j (respectively, f_i and f_j) for $\mathcal{B}$ commute (setting $e_i(0) = f_i(0) = 0$).

Proof. This holds because the crystal graph of $\mathcal{B}_J$ for $J = \{i, j\}$ is a rectangle. □

Lemma 1.5. If $\mathcal{B}$ is a Stembridge crystal then the dual $\mathcal{B}^\vee$ is a Stembridge crystal for the same type.

Proof. This property will be checked in HW2. □

2 Root system embeddings

The theme of our next few lectures will be generalizing Stembridge crystals to non-simply-laced types. A key ingredient in the relevant constructions will be certain embeddings of non-simply-laced weight lattices inside simply-laced weight lattices.

Suppose we have an embedding of Lie algebras $X \hookrightarrow Y$. Then there is a natural restriction map $\Lambda^Y \to \Lambda^X$ of the associated weight lattices. We are interested in the adjoint $\Lambda^X \hookrightarrow \Lambda^Y$ of this map.

It is enough to consider $X \hookrightarrow Y$ in the following cases, in which Y is simply-laced but X is not:

(1) $X = C_n \hookrightarrow A_{2n-1} = Y$,

(2) $X = B_n \hookrightarrow D_{n+1} = Y$,

(3) $X = F_4 \hookrightarrow E_6 = Y$, or

(4) $X = G_2 \hookrightarrow D_4 = Y$.

Let Λ^X, Λ^Y be the corresponding weight lattices and let Φ^X, Φ^Y be the corresponding root systems. Today, we always assume each weight lattice is semisimple. This means that both the fundamental weights and the simple roots form a $\mathbb{Z}$-basis for each weight lattice.

We write α_i^X, α_i^Y and ϖ_i^X, ϖ_i^Y for the corresponding simple roots and fundamental weights.

Let I^X and I^Y be the indexing sets of the simple roots / fundamental weights in each case. If X has rank m and Y has rank n then we will always have $I^X = \{1, 2, \dots, m\}$ and $I^Y = \{1, 2, \dots, n\}$.

The embedding of weight lattices we wish to describe will be a linear map $\Psi : \Lambda^X \to \Lambda^Y$ satisfying

$$\boxed{\Psi(\varpi_i^X) = \gamma_i \sum_{j \in \sigma(i)} \varpi_j^Y \qquad \text{and} \qquad \Psi(\alpha_i^X) = \gamma_i \sum_{j \in \sigma(i)} \alpha_j^Y \qquad \text{for } i \in I^X} \tag{2.1}$$

where γ_i are certain positive integers and σ is a certain map $I^X \to \{$ subsets of I^Y $\}$.

The map σ will always have the following properties:

- For each $i \in I^X$, the simple roots indexed by $j \in \sigma(i)$ are mutually orthogonal.
- The sets $\sigma(i)$ for $i \in I^X$ are disjoint and $I^Y = \bigsqcup_{i\in I^X} \sigma(i)$.

Briefly, here are the specific values of the relevant parameters:

(1) Suppose $(X, Y) = (C_n, A_{2n-1})$. Then the simple roots are

$$\begin{aligned} \alpha_1^X &= \mathbf{e}_1 - \mathbf{e}_2 \\ \alpha_2^X &= \mathbf{e}_2 - \mathbf{e}_3 \\ &\vdots \\ \alpha_{n-1}^X &= \mathbf{e}_{n-1} - \mathbf{e}_n \\ \alpha_n^X &= 2\mathbf{e}_n \end{aligned} \qquad \text{and} \qquad \begin{aligned} \alpha_1^Y &= \mathbf{e}_1 - \mathbf{e}_2 \\ \alpha_2^Y &= \mathbf{e}_2 - \mathbf{e}_3 \\ \alpha_3^Y &= \mathbf{e}_3 - \mathbf{e}_4 \\ &\vdots \\ \alpha_{2n-1}^Y &= \mathbf{e}_{2n-1} - \mathbf{e}_{2n} \end{aligned}$$

and we set $\boxed{\gamma_1 = \gamma_2 = \cdots = \gamma_{n-1} = 1,\ \gamma_n = 2, \text{ and } \sigma(i) = \{i, 2n-i\} \text{ for } 1 \leq i \leq n.}$

The fundamental weights in this case are

$$\begin{aligned} \varpi_1^X &= \mathbf{e}_1 \\ \varpi_2^X &= \mathbf{e}_1 + \mathbf{e}_2 \\ &\vdots \\ \varpi_{n-1}^X &= \mathbf{e}_1 + \cdots + \mathbf{e}_{n-1} \\ \varpi_n^X &= \mathbf{e}_1 + \cdots + \mathbf{e}_n \end{aligned} \qquad \text{and} \qquad \begin{aligned} \varpi_1^Y &= \mathbf{e}_1 \\ \varpi_2^Y &= \mathbf{e}_1 + \mathbf{e}_2 \\ \varpi_3^Y &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \\ &\vdots \\ \varpi_{2n-1}^Y &= \mathbf{e}_1 + \cdots + \mathbf{e}_{2n-1}. \end{aligned}$$

Since we are assuming that our weight lattices are semisimple, the ambient vector space for A_{2n-1} is the quotient vector space $V = \mathbb{R}^{2n}/\mathbb{R}\text{-span}\{\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_{2n}\}$.

(2) Suppose $(X, Y) = (B_n, D_{n+1})$. Then the simple roots are

$$\begin{aligned} \alpha_1^X &= \mathbf{e}_1 - \mathbf{e}_2 \\ \alpha_2^X &= \mathbf{e}_2 - \mathbf{e}_3 \\ &\vdots \\ \alpha_{n-1}^X &= \mathbf{e}_{n-1} - \mathbf{e}_n \\ \alpha_n^X &= \mathbf{e}_n \end{aligned} \qquad \text{and} \qquad \begin{aligned} \alpha_1^Y &= \mathbf{e}_1 - \mathbf{e}_2 \\ \alpha_2^Y &= \mathbf{e}_2 - \mathbf{e}_3 \\ &\vdots \\ \alpha_n^Y &= \mathbf{e}_n - \mathbf{e}_{n+1} \\ \alpha_{n+1}^Y &= \mathbf{e}_n + \mathbf{e}_{n+1} \end{aligned}$$

and we set $\boxed{\gamma_i = 2 \text{ and } \sigma(i) = \{i\} \text{ for } 1 \leq i < n, \text{ along with } \gamma_n = 1 \text{ and } \sigma(n) = \{n, n+1\}.}$

The fundamental weights in this case are

$$\begin{aligned} \varpi_1^X &= \mathbf{e}_1 \\ \varpi_2^X &= \mathbf{e}_1 + \mathbf{e}_2 \\ &\vdots \\ \varpi_{n-1}^X &= \mathbf{e}_1 + \cdots + \mathbf{e}_{n-1} \\ \varpi_n^X &= \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \varpi_1^Y &= \mathbf{e}_1 \\ \varpi_2^Y &= \mathbf{e}_1 + \mathbf{e}_2 \\ &\vdots \\ \varpi_n^Y &= \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n - \mathbf{e}_{n+1}) \\ \varpi_{n+1}^Y &= \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n + \mathbf{e}_{n+1}). \end{aligned}$$

(3) Suppose $(X,Y) = (F_4, E_6)$. Then the simple roots are

$$\begin{aligned} \alpha_1^X &= \mathbf{e}_1 - \mathbf{e}_2 \\ \alpha_2^X &= \mathbf{e}_2 - \mathbf{e}_3 \\ \alpha_3^X &= \mathbf{e}_3 \\ \alpha_4^X &= -\tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) \end{aligned} \quad \text{and} \quad \begin{aligned} \alpha_1^Y &= \mathbf{e}_1 - \mathbf{e}_2 \\ \alpha_2^Y &= \mathbf{e}_2 - \mathbf{e}_3 \\ \alpha_3^Y &= \mathbf{e}_3 - \mathbf{e}_4 \\ \alpha_4^Y &= \mathbf{e}_4 - \mathbf{e}_5 \\ \alpha_5^Y &= \mathbf{e}_4 + \mathbf{e}_5 \\ \alpha_6^Y &= -\tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8). \end{aligned}$$

and we set $\boxed{\gamma_1 = \gamma_2 = 2,\ \gamma_3 = \gamma_4 = 1 \text{ and } \sigma(1) = \{4\},\ \sigma(2) = \{3\},\ \sigma(3) = \{2,5\},\ \sigma(4) = \{1,6\}.}$

The fundamental weights in this case are

$$\begin{aligned} \varpi_1^X &= \mathbf{e}_1 - \mathbf{e}_4 \\ \varpi_2^X &= \mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_4 \\ \varpi_3^X &= \tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 - 3\mathbf{e}_4) \\ \varpi_4^X &= -\mathbf{e}_4 \end{aligned} \quad \text{and} \quad \begin{aligned} \varpi_1^Y &= \mathbf{e}_1 - \tfrac{1}{3}(\mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8) \\ \varpi_2^Y &= \mathbf{e}_1 + \mathbf{e}_2 - \tfrac{2}{3}(\mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8) \\ \varpi_3^Y &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 - (\mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8) \\ \varpi_4^Y &= \tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6 - \mathbf{e}_7 - \mathbf{e}_8) \\ \varpi_5^Y &= \tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 - \tfrac{5}{3}\mathbf{e}_6 - \tfrac{5}{3}\mathbf{e}_7 - \tfrac{5}{3}\mathbf{e}_8) \\ \varpi_6^Y &= -\tfrac{2}{3}(\mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8). \end{aligned}$$

Since we are assuming that our weight lattices are semisimple, the ambient vector space for E_6 is the quotient vector space $V = \mathbb{R}^8/\mathbb{R}\text{-span}\{\mathbf{e}_6 - \mathbf{e}_7, \mathbf{e}_7 - \mathbf{e}_8\}$.

(4) Suppose $(X,Y) = (G_2, D_4)$. Then the simple roots are

$$\begin{aligned} \alpha_1^X &= \mathbf{e}_1 - \mathbf{e}_2 \\ \alpha_2^X &= -\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3 \end{aligned} \quad \text{and} \quad \begin{aligned} \alpha_1^Y &= \mathbf{e}_1 - \mathbf{e}_2 \\ \alpha_2^Y &= \mathbf{e}_2 - \mathbf{e}_3 \\ \alpha_3^Y &= \mathbf{e}_3 - \mathbf{e}_4 \\ \alpha_4^Y &= \mathbf{e}_3 + \mathbf{e}_4 \end{aligned}$$

and we set $\boxed{\gamma_1 = 1 \text{ and } \gamma_2 = 3 \text{ along with } \sigma(1) = \{1,3,4\}, \text{ and } \sigma(2) = \{2\}.}$

The fundamental weights in this case are

$$\begin{aligned} \varpi_1^X &= \mathbf{e}_1 - \mathbf{e}_3 \\ \varpi_2^X &= \mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3 \end{aligned} \quad \text{and} \quad \begin{aligned} \varpi_1^Y &= \mathbf{e}_1 \\ \varpi_2^Y &= \mathbf{e}_1 + \mathbf{e}_2 \\ \varpi_3^Y &= \tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4) \\ \varpi_4^Y &= \tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4). \end{aligned}$$

Since we are assuming that our weight lattices are semisimple, the ambient vector space for G_2 is the quotient vector space $V = \mathbb{R}^3/\mathbb{R}\text{-span}\{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$.

Example 2.1. If $(X,Y) = (C_2, A_3)$ then $\gamma_1 = 1$, $\gamma_2 = 2$, $\sigma(1) = \{1,3\}$, and $\sigma(2) = \{2\}$, so

$$\begin{aligned} \Psi(\mathbf{e}_1) = \Psi(\varpi_1^{C_2}) &= \varpi_1^{A_3} + \varpi_3^{A_3} = 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \\ \Psi(\mathbf{e}_1 + \mathbf{e}_2) = \Psi(\varpi_2^{C_2}) &= 2\varpi_2^{A_3} = 2\mathbf{e}_1 + 2\mathbf{e}_2 \end{aligned}$$

while

$$\begin{aligned} \Psi(\mathbf{e}_1 - \mathbf{e}_2) = \Psi(\alpha_1^{C_2}) &= \alpha_1^{A_3} + \alpha_3^{A_3} = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4 \\ \Psi(2\mathbf{e}_2) = \Psi(\alpha_2^{C_2}) &= 2\alpha_2^{A_3} = 2\mathbf{e}_2 - 2\mathbf{e}_3. \end{aligned}$$

These identities imply $\Psi(\mathbf{e}_1) = 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = \mathbf{e}_1 - \mathbf{e}_4$ which is consistent in $\mathbb{R}^4/\mathbb{R}\text{-span}\{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4\}$.

3 Virtual crystals

Continue to let $X \hookrightarrow Y$ be an embedding of Lie algebras described by (1), (2), (3), or (4).

Note that this means that Y is of simply-laced type.

Our next task is to use the embedding $\Psi : \Lambda^X \to \Lambda^Y$ to construct certain crystals of non-simply-laced type X that will be our analogues of Stembridge crystals.

For now, we still assume the weight lattices Λ^X and Λ^Y are semisimple.

Let $\widehat{\mathcal{V}}$ be a Stembridge crystal of type Y with weight map $\widehat{\mathbf{wt}} : \widehat{\mathcal{V}} \to \Lambda^Y$, crystal operators $\widehat{e}_i$, $\widehat{f}_i$, and string lengths $\widehat{\varepsilon}_i$, $\widehat{\varphi}_i$ for $i \in I^Y$. We call this structure the *ambient crystal*.

Define *virtual crystal operators* (of type X) for $i \in I^X$ by the formulas

$$e_i := \prod_{j\in\sigma(i)} (\widehat{e}_j)^{\gamma_i} \qquad \text{and} \qquad f_i := \prod_{j\in\sigma(i)} \left(\widehat{f}_j\right)^{\gamma_i}$$

where $\sigma : I^X \to \{ \text{ subsets of } I^Y \ \}$ and $\gamma_i \in \mathbb{N}$ correspond to (X, Y) as in the previous section.

These products give well-defined maps $\widehat{\mathcal{V}} \to \widehat{\mathcal{V}} \sqcup \{0\}$, regardless of the order in which they are evaluated, as a consequence of Lemma 1.4. Also define the *virtual string lengths* (of type X) for $i \in I^X$ by

$$\varepsilon_i(b) = \frac{1}{\gamma_i} \cdot \frac{1}{|\sigma(i)|} \sum_{j\in\sigma(i)} \widehat{\varepsilon}_j(b) \qquad \text{and} \qquad \varphi_i(b) = \frac{1}{\gamma_i} \cdot \frac{1}{|\sigma(i)|} \sum_{j\in\sigma(i)} \widehat{\varphi}_j(b) \qquad \text{for } b \in \widehat{\mathcal{V}}.$$

Since $\widehat{\mathcal{V}}$ is seminormal, the virtual string lengths could have nonnegative *rational* values.

Definition 3.1. A subset $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ is a *virtual crystal* if for each $b \in \mathcal{V}$ and $i \in I^X$ the following holds:

(V1) The string lengths $\widehat{\varepsilon}_j(b)$ and $\widehat{\varphi}_j(b)$ have the same values for all $j \in \sigma(i)$ and these values are multiples of γ_i. Consequently $\varepsilon_i(b) = \frac{1}{\gamma_i}\widehat{\varepsilon}_j(b) \in \mathbb{N}$ and $\varphi_i(b) = \frac{1}{\gamma_i}\widehat{\varphi}_j(b) \in \mathbb{N}$ for any $j \in \sigma(i)$.

(V2) The virtual crystal operators e_i and f_i restrict to maps $\mathcal{V} \to \mathcal{V} \sqcup \{0\}$ and we have

$$\varepsilon_i(b) = \max\{k \geq 0 : e_i^k(b) \neq 0\} \qquad \text{and} \qquad \varphi_i(b) = \max\{k \geq 0 : f_i^k(b) \neq 0\}.$$

Suppose $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ is a virtual crystal. Define $\mathbf{wt} : \mathcal{V} \to \Lambda^X$ by

$$\mathbf{wt}(b) := \sum_{i\in I^X} (\varphi_i(b) - \varepsilon_i(b))\, \varpi_i^X.$$

Proposition 3.2. It holds that $\Psi(\mathbf{wt}(b)) = \widehat{\mathbf{wt}}(b)$ for all $b \in \mathcal{V}$.

Proof. If $b \in \mathcal{V}$ then, using axiom (V1), we have

$$\begin{aligned} \Psi(\mathbf{wt}(b)) &= \sum_{i\in I^X} (\varphi_i(b) - \varepsilon_i(b))\, \Psi(\varpi_i^X) \\ &= \sum_{i\in I^X} \sum_{j\in\sigma(i)} \gamma_i\, (\varphi_i(b) - \varepsilon_i(b))\, \varpi_j^Y = \sum_{i\in I^X} \sum_{j\in\sigma(i)} (\widehat{\varphi}_j(b) - \widehat{\varepsilon}_j(b))\, \varpi_j^Y. \end{aligned}$$

The summation $\sum_{i\in I^X} \sum_{j\in\sigma(i)}$ is the same as $\sum_{j\in I^Y}$ so the last expression is equal to the weight map of $\widehat{\mathcal{V}}$ evaluated at b by a result in Lecture 5 (which requires Λ^Y to semisimple). □

Proposition 3.3. Continue to let $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ be a virtual crystal. Then $\mathcal{V}$ is a seminormal crystal of type X relative to the operators $\mathbf{wt}$, e_i, f_i, ε_i, φ_i.

Proof. Suppose $x, y \in \mathcal{V}$ and $i \in I^X$. The first axiom we need to check from the definition of a crystal is that $e_i(x) = y$ if and only if $f_i(y) = x$. This holds because the same axiom applied (repeatedly) to $\widehat{\mathcal{V}}$ implies that $y = \prod_{j\in\sigma(i)} \widehat{e}_j^{\gamma_i}(x)$ if and only if $x = \prod_{j\in\sigma(i)} \widehat{f}_j^{\gamma_i}(y)$. In this event, we have

$$\varepsilon_i(y) = \tfrac{1}{\gamma_i}\widehat{\varepsilon}_j(y) = \tfrac{1}{\gamma_i}\left(\widehat{\varepsilon}_j(x) - \gamma_i\right) = \varepsilon_i(x) - 1$$

and it follows similarly that $\varphi_i(y) = \varphi_i(x) + 1$; moreover, it holds that

$$\Psi(\mathbf{wt}(y)) = \widehat{\mathbf{wt}}(y) = \widehat{\mathbf{wt}}(x) + \gamma_i \sum_{j\in\sigma(i)} \alpha_j^Y = \widehat{\mathbf{wt}}(x) + \Psi(\alpha_i^X) = \Psi(\mathbf{wt}(x) + \alpha_i^X)$$

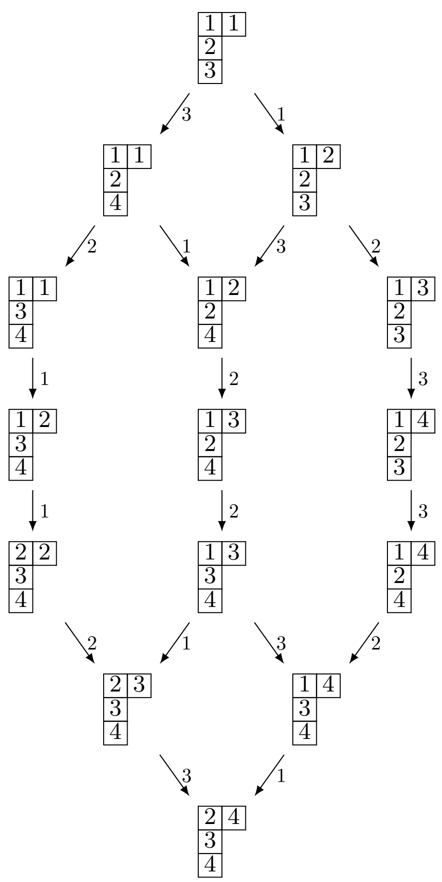
so $\mathbf{wt}(y) = \mathbf{wt}(x) + \alpha_i^X$ since Ψ is injective. This confirms the first crystal axiom for $\mathcal{V}$.

To check the second crystal axiom it suffices to show that $\varphi_i(x) - \varepsilon_i(x) = \langle \mathbf{wt}(x), \alpha_i^\vee \rangle$. But this is immediate from the definition of $\mathbf{wt}(x)$ since the fundamental weights and coroots are dual bases (when the weight lattices are semimple, as we are currently assuming).

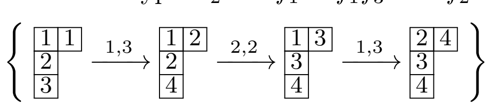
Thus $\mathcal{V}$ is a crystal of type X. This crystal is seminormal by condition (V2). □

Example 3.4. We consider the embedding $C_2 \hookrightarrow A_3$ and take $\widehat{\mathcal{V}} = \mathrm{SSYT}_4(\lambda)$ for $\lambda = (2, 1, 1)$.

The type A_3 lowering operators are $\widehat{f}_1, \widehat{f}_2, \widehat{f}_3$ and the crystal graph is shown below:



The lowering operators for type C_2 are $f_1 = \widehat{f}_1\widehat{f}_3$ and $f_2 = \widehat{f}_2\widehat{f}_2$.

The subset $\mathcal{V} =$  is a virtual crystal.

This crystal $\mathcal{V}$ is isomorphic to the standard crystal for type C_2.

For each of the four embeddings $X \hookrightarrow Y$, we defined an associated map $\sigma : I^X \to \{ \text{ subsets of } I^Y \ \}$.

Each subset in the image of σ is an orbit under a certain permutation $\mathsf{aut} : I^Y \to I^Y$ that induces an automorphism of the root system Φ^Y and weight lattice Λ^Y, which we also denote by aut. Specifically:

(1) If the embedding is $C_n \hookrightarrow A_{2n-1}$, then aut is the map $i \mapsto 2n - i$.

(2) If the embedding is $B_n \hookrightarrow D_{n+1}$, then aut interchanges $n \leftrightarrow n+1$ while fixing $1 \leq i < n$.

(3) If the embedding is $F_4 \hookrightarrow E_6$ then aut interchanges $1 \leftrightarrow 6$ and $2 \leftrightarrow 5$ while fixing 3 and 4.

(4) If the embedding is $G_2 \hookrightarrow D_4$ then aut maps $1 \mapsto 4 \mapsto 3 \mapsto 1$ while fixing 2.

Proposition 3.5. Suppose $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ is a virtual crystal for the embedding $X \hookrightarrow Y$.

(a) Any highest weight element $u \in \mathcal{V}$ is a highest weight element of $\widehat{\mathcal{V}}$ and $\widehat{\mathbf{wt}}(u) = \mathsf{aut}\left(\widehat{\mathbf{wt}}(u)\right) \in \Lambda^Y$.

(b) If $\mathcal{V}$ and $\widehat{\mathcal{V}}$ are both connected, then there exists a bijection $\widehat{\mathcal{V}} \to \widehat{\mathcal{V}}$, which becomes an automorphism of the crystal graph after we permute the edge labels by aut, that fixes every element of $\mathcal{V}$.

Remark. In the example on the previous page, the bijection $\widehat{\mathcal{V}} \to \widehat{\mathcal{V}}$ described by this result is the map that flips the displayed crystal graph of $\widehat{\mathcal{V}} = \mathrm{SSYT}_4(\lambda)$ across its central vertical axis. From this example, we see that not all fixed points are necessarily elements of the virtual crystal $\mathcal{V}$; consider $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$.

Proof. An element $u \in \mathcal{V}$ is a highest weight element if and only if $\varepsilon_i(u) = 0$ for all $i \in I^X$, but axiom (V1) implies that this holds if and only if $\widehat{\varepsilon}_j(u) = 0$ for all $j \in I^Y$. In this case, by Proposition 3.2, we have $\widehat{\mathbf{wt}}(u) = \Psi(\mathbf{wt}(u))$, so $\widehat{\mathbf{wt}}(u) = \mathsf{aut}\left(\widehat{\mathbf{wt}}(u)\right)$ since $\mathsf{aut} \circ \Psi = \Psi$.

Assume $\mathcal{V}$ and $\widehat{\mathcal{V}}$ are connected. Since $\widehat{\mathcal{V}}$ is a Stembridge crystal, it has a unique highest weight element u. By the first paragraph, this element must also be the unique highest weight of $\mathcal{V}$. Permuting the labels of the edges in the crystal graph of $\widehat{\mathcal{V}}$ gives a new Stembridge crystal with highest weight $\mathsf{aut}\left(\widehat{\mathbf{wt}}(u)\right)$.

Since this is equal to $\widehat{\mathbf{wt}}(u)$, there must exist an isomorphism between the new crystal and the original crystal. This isomorphism induces a bijection $\widehat{\mathcal{V}} \to \widehat{\mathcal{V}}$ of crystal graphs which must fix the unique highest weight element u. It is clear from the definitions that any element of $\mathcal{V}$ derived from u by an application of the virtual crystal operators is also a fixed point of the bijection $\widehat{\mathcal{V}} \to \widehat{\mathcal{V}}$. $\square$