1 Last time: Root system embeddings and virtual crystals

Assume $X \hookrightarrow Y$ is one of the following embeddings of Lie algebras / root systems / Cartan types:

(1) $X = C_n \hookrightarrow A_{2n-1} = Y$,

(2) $X = B_n \hookrightarrow D_{n+1} = Y$,

(3) $X = F_4 \hookrightarrow E_6 = Y$, or

(4) $X = G_2 \hookrightarrow D_4 = Y$.

Let Λ^X, Λ^Y be the corresponding semisimple weight lattices and let Φ^X, Φ^Y be the corresponding root systems. Write α_i^X, α_i^Y and ϖ_i^X, ϖ_i^Y for the corresponding simple roots and fundamental weights.

Let I^X and I^Y be the indexing sets for the simple roots (and fundamental weights) in each case.

Last time, we specified a particular injective linear map $\Psi : \Lambda^X \to \Lambda^Y$ given by the formula

$$\boxed{\Psi(\varpi_i^X) = \gamma_i \sum_{j\in\sigma(i)} \varpi_j^Y \qquad \text{and} \qquad \Psi(\alpha_i^X) = \gamma_i \sum_{j\in\sigma(i)} \alpha_j^Y \qquad \text{for } i \in I^X} \tag{1.1}$$

where γ_i are positive integers and σ is a map $I^X \to \{$ nonempty subsets of I^Y $\}$ with $I^Y = \bigsqcup_{i\in I^X} \sigma(i)$.

We then gave a definition of *virtual crystals* of type X in terms of γ_i and σ.

Specifically, let $\widehat{\mathcal{V}}$ be a Stembridge crystal of type Y with weight map $\widehat{\mathbf{wt}} : \widehat{\mathcal{V}} \to \Lambda^Y$, crystal operators $\widehat{e}_i$, $\widehat{f}_i$, and string lengths $\widehat{\varepsilon}_i$, $\widehat{\varphi}_i$ for $i \in I^Y$. We call $\widehat{\mathcal{V}}$ the *ambient crystal*. Define *virtual crystal operators*

$$\boxed{e_i := \prod_{j\in\sigma(i)} (\widehat{e}_j)^{\gamma_i}} \qquad \text{and} \qquad \boxed{f_i := \prod_{j\in\sigma(i)} \left(\widehat{f}_j\right)^{\gamma_i}} \qquad \text{for } i \in I^X.$$

Definition 1.1. A subset $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ is a *virtual crystal* if for each $b \in \mathcal{V}$ and $i \in I^X$ the following holds:

(V1) The string length $\widehat{\varepsilon}_j(b)$ has the same value for all $j \in \sigma(i)$ and this value is a multiple of γ_i.

The string length $\widehat{\varphi}_j(b)$ has the same value for all $j \in \sigma(i)$ and this value is a multiple of γ_i.

This lets us define $\boxed{\varepsilon_i(b) := \frac{1}{\gamma_i}\widehat{\varepsilon}_j(b) \in \mathbb{N}}$ and $\boxed{\varphi_i(b) := \frac{1}{\gamma_i}\widehat{\varphi}_j(b) \in \mathbb{N}}$ for any $j \in \sigma(i)$.

(V2) The virtual crystal operators e_i and f_i restrict to maps $\mathcal{V} \to \mathcal{V} \sqcup \{0\}$ and we have

$$\varepsilon_i(b) = \max\{k \geq 0 : e_i^k(b) \neq 0\} \qquad \text{and} \qquad \varphi_i(b) = \max\{k \geq 0 : f_i^k(b) \neq 0\}.$$

Assume $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ is a virtual crystal and define $\mathbf{wt} : \mathcal{V} \to \Lambda^X$ by $\boxed{\mathbf{wt}(b) := \sum_{i\in I^X} (\varphi_i(b) - \varepsilon_i(b))\, \varpi_i^X}$.

We showed last time that $\Psi(\mathbf{wt}(b)) = \widehat{\mathbf{wt}}(b)$ for all $b \in \mathcal{V}$ and that $\mathcal{V}$ is a seminormal crystal of type X.

The subsets in the image of the map σ are the orbits of a certain permutation $\mathsf{aut} : I^Y \to I^Y$ that induces an automorphism of the root system Φ^Y and weight lattice Λ^Y, which we also denote by aut.

If $\mathcal{V}$ and $\widehat{\mathcal{V}}$ are both connected, then there exists a bijection $\widehat{\mathcal{V}} \to \widehat{\mathcal{V}}$, which becomes an automorphism of the crystal graph after we permute the edge labels by aut, that fixes every element of $\mathcal{V}$.

Finally, any highest weight element $u \in \mathcal{V}$ is a highest weight element of $\widehat{\mathcal{V}}$ and $\widehat{\mathbf{wt}}(u) = \mathsf{aut}\left(\widehat{\mathbf{wt}}(u)\right)$.

2 Properties of virtual crystals

We start today by leveraging our setup from last time to prove three "Stembridge-like" properties of virtual crystals. Continue to let $X \hookrightarrow Y$ be one of our four Lie algebra embeddings.

Theorem 2.1. Suppose $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ and $\mathcal{W} \subseteq \widehat{\mathcal{W}}$ are virtual crystals for the embedding $X \hookrightarrow Y$.

Then $\mathcal{V} \otimes \mathcal{W} \subseteq \widehat{\mathcal{V}} \otimes \widehat{\mathcal{W}}$ is also a virtual crystal for the embedding $X \hookrightarrow Y$.

Proof. We may assume that all crystals here are connected. The tensor product of Stembridge crystals $\widehat{\mathcal{V}} \otimes \widehat{\mathcal{W}}$ remains a Stembridge crystal, so we just need to check axioms (V1) and (V2).

Fix $i \in I^X$, $v \in \mathcal{V}$, and $w \in \mathcal{W}$.

Since the automorphism aut fixes each element of $\mathcal{V}$ and $\mathcal{W}$ while permuting transitively the edge labels $j \in \sigma(i)$, it follows that aut also induces an automorphism of the crystal graph of $\widehat{\mathcal{V}} \otimes \widehat{\mathcal{W}}$ after permuting edge labels. Thus $\widehat{\varepsilon}_j(v \otimes w)$ and $\widehat{\varphi}_j(v \otimes w)$ take the same value for all $j \in \sigma(i)$.

Since $\widehat{\mathcal{V}}$ and $\widehat{\mathcal{W}}$ are finite type crystals, the definition of $\otimes$ gives

$$\widehat{\varphi}_j(v \otimes w) = \widehat{\varphi}_j(v) + \max\{0, \widehat{\varphi}_j(w) - \widehat{\varepsilon}_j(v)\}$$

for any $j \in I^Y$. By axiom (V1) applied to $\mathcal{V}$ and $\mathcal{W}$, this means that

$$\widehat{\varphi}_j(v \otimes w) = \gamma_i \left(\varphi_i(v) + \max\{0, \varphi_i(w) - \varepsilon_i(v)\}\right)$$

whenever $j \in \sigma(i)$. The right side is γ_i times the formula for $\varphi_i(v \otimes w)$ from the definition of $\otimes$, so we get $\gamma_i \varphi_i(v \otimes w) = \widehat{\varphi}_j(v \otimes w)$ for any $j \in \sigma(i)$ as needed.

It follows similarly that $\gamma_i \varepsilon_i(v \otimes w) = \widehat{\varepsilon}_j(v \otimes w)$ for any $j \in \sigma(i)$, so (V1) holds.

Since tensor products of seminormal crystals are seminormal, to check axiom (V2) it suffices to show that $f_i(v \otimes w)$ is either $f_i(v) \otimes w$ or $v \otimes f_i(w)$, and that $e_i(v \otimes w)$ is either $e_i(v) \otimes w$ or $v \otimes e_i(w)$. We just prove the first property since the argument for e_i is similar.

If $\gamma_i = 1$ then f_i is a product of distinct commuting crystal operators $\widehat{f}_j$ so the desired claim is easy to check directly. Assume $\gamma_i > 1$. After reviewing our definitions of γ_i and σ from last lecture, one finds that in this case $\sigma(i) = \{j\}$ always consists of a single element. Therefore

$$f_i(v \otimes w) = (\widehat{f}_j)^{\gamma_i}(v \otimes w) = \begin{cases} (\widehat{f}_j)^{\gamma_i}(v) \otimes w & \text{if } \widehat{\varphi}_j(w) \leq \widehat{\varepsilon}_j(v), \\ v \otimes (\widehat{f}_j)^{\gamma_i}(w) & \text{if } \widehat{\varphi}_j(w) \geq \widehat{\varepsilon}_j(v) + \gamma_i. \end{cases}$$

We cannot have $\widehat{\varepsilon}_j(v) < \widehat{\varphi}_j(w) < \widehat{\varepsilon}_j(v) + \gamma_i$ since $\widehat{\varepsilon}_j(v)$ and $\widehat{\varphi}_j(w)$ are multiples of γ_i by axiom (V1). Thus $f_i(v \otimes w) \in \{f_i(v) \otimes w, v \otimes f_i(w)\}$ as needed and we conclude that (V2) holds for $\mathcal{V} \otimes \mathcal{W}$. □

Theorem 2.2. Suppose $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ is a virtual crystal for the embedding $X \hookrightarrow Y$.

If $\mathcal{V}$ is connected (as a crystal of type X) then it has a unique highest weight element.

Proof. The connected virtual crystal $\mathcal{V}$ is contained in a full subcrystal the Stembridge crystal $\widehat{\mathcal{V}}$, which has a unique highest weight element. This must also be the unique highest weight element of $\mathcal{V}$ since any highest weight element for $\mathcal{V}$ is a highest weight element for $\widehat{\mathcal{V}}$. □

Theorem 2.3. Suppose $\mathcal{V}, \mathcal{V}' \subseteq \widehat{\mathcal{V}}$ are two connected virtual crystals whose unique highest weight elements have the same weight. Then $\mathcal{V}$ and $\mathcal{V}'$ are isomorphic as type X crystals.

Proof. Let $u \in \mathcal{V}$ and $u' \in \mathcal{V}'$ be the relevant highest weight elements.

Since $\mathbf{wt}(u) = \mathbf{wt}(u') \in \Lambda^X$ we also have $\widehat{\mathbf{wt}}(u) = \Psi(\mathbf{wt}(u)) = \Psi(\mathbf{wt}(u')) = \widehat{\mathbf{wt}}(u') \in \Lambda^Y$.

The elements u and u' are highest weights in the Stembridge crystal $\widehat{\mathcal{V}}$, so they belong to isomorphic (Stembridge) full subcrystals.

This means that $\mathcal{V}$ and $\mathcal{V}'$ are generated by elements of equal weight in isomorphic full subcrystals with the same crystal operators, so we must have $\mathcal{V} \cong \mathcal{V}'$. □

3 Fundamental crystals

We turn our attention to the classical Cartan types A_n, B_n, C_n, and D_n.

Assume Φ is a root system of one of these types and Λ is a corresponding weight lattice. As usual write $\{\alpha_i : i \in I\}$ and $\{\varpi_i : i \in I\}$ for the simple roots and fundamental weights. We assume Λ is *simply-connected* is the sense that $\Lambda = \mathbb{R}\text{-span}\{\varpi_i : i \in I\}$.

We have $\varpi_i = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_i$ in all cases except in type B_n when $i = n$ or type D_n when $i \in \{n-1, n\}$.

In type B_n one has $\varpi_n = \frac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n)$.

In type D_n, one has $\varpi_{n-1} = \frac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n-1} - \mathbf{e}_n)$ and $\varpi_n = \frac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n-1} + \mathbf{e}_n)$.

We refer to these exceptions as *spin fundamental weights*.

Let $\mathbb{B}$ be the standard crystal associated to (Φ, Λ) and recall that an element b of a crystal is a highest weight element of $e_i(b) = 0$ for all i.

Proposition 3.1. Each full subcrystal of the tensor power $\mathbb{B}^{\otimes m}$ has a unique highest weight element.

Proof. In types A_n and D_n, this holds because $\mathbb{B}$ is a Stembridge crystal.

In the remaining types, by the theorems in the previous section, it suffices to show that $\mathbb{B}$ is isomorphic to a virtual crystal for the appropriate root system embedding.

In type C_n, the standard crystal $\mathbb{B}$ can be realized as a virtual crystal for the embedding $C_n \hookrightarrow A_{2n-1}$ by taking $\widehat{\mathcal{V}} = \mathrm{SSYT}_{2n}(\lambda)$ for $\lambda = 1^{2n-2}2$ and letting $\mathcal{V}$ be generated by the highest weight element

$$T = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} 2n-1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \in \mathrm{SSYT}_{2n}(\lambda)$$

under the crystal operators $f_i = \widehat{f}_i \widehat{f}_{2n-i}$ for $1 \le i < n$ and $f_n = (\widehat{f}_n)^2$.

Applying $f_i \cdots f_2 f_1$ to T has the effect of adding 1 to the last i boxes in the first column and adding i to the second box in the first row. Applying $f_{n-i} \cdots f_{n-2} f_{n-1}$ to $f_n \cdots f_2 f_1(T)$ then continues to add 1 to successive boxes in the first column while also adding to the second box in the first row. We discussed this construction for $n = 2$ in the previous lecture.

In type B_n, the standard crystal $\mathbb{B}$ can be realized as a virtual crystal for the embedding $B_n \hookrightarrow D_{n+1}$ by taking $\widehat{\mathcal{V}}$ to be the tensor product two copies of the standard crystal for type D_{n+1} and letting $\mathcal{V}$ be generated by the highest weight element $\boxed{1} \otimes \boxed{1}$ under $f_i = (\widehat{f}_i)^2$ for $1 \leq i < r$ and $f_r = \widehat{f}_r \widehat{f}_{r+1}$.

See Figure 5.4 in Bump and Schilling's book for an example of this construction when $n = 2$.

More details are needed to thoroughly check that the virtual crystals just described are in fact isomorphic to the standard crystals in types B_n and C_n. But this is a completely straightforward exercise, using the signature rule in type $C_n \hookrightarrow A_{2n-1}$ or the definition of the tensor product in type $B_n \hookrightarrow D_{n+1}$. □

Our next goal is to construct a "fundamental crystal" $\mathcal{B}_{\varpi_i}$ with unique highest weight ϖ_i for each $i \in I$.

This is accomplished for the first fundamental weight by setting $\mathcal{B}_{\varpi_1} = \mathbb{B}$, since in all four classical types, the crystal $\mathbb{B}$ itself has unique highest weight element $\boxed{1}$ with weight $\varpi_1 = \mathbf{e}_1$.

The following proposition covers almost all of the remaining weights:

Proposition 3.2. If $k \in \{1, 2, \dots, n\}$ and ϖ_k is not a spin fundamental weight in types B_n or D_n, then $\mathbb{B}^{\otimes k}$ has a full subcrystal with a unique highest weight element of weight $\varpi_k = \mathbf{e}_1 + \cdots + \mathbf{e}_k$.

Proof. We need to exhibit $u \in \mathbb{B}^{\otimes k}$ with $\mathbf{wt}(u) = \varpi_k = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_k$ and $e_i(u) = 0$ for all $i \in I$.

Since we dealing with seminormal crystals, the second condition is equivalent to $\varepsilon_i(u) = 0$ for all $i \in I$.

The element $u = \boxed{k} \otimes \cdots \otimes \boxed{2} \otimes \boxed{1} \in \mathbb{B}^{\otimes k}$ has the right weight, and one can show by induction that

$$\varepsilon_i(u) = \max_{1 \leq j \leq k} \left(\sum_{h=1}^{j} \varepsilon_i(\boxed{h}) - \sum_{h=1}^{j-1} \varphi_i(\boxed{h}) \right).$$

Since $\varepsilon_i(\boxed{1}) = 0$ we can rewrite this as $\varepsilon_i(u) = \max_{1 \leq j \leq k} \left(\sum_{h=1}^{j-1} \left(\varepsilon_i(\boxed{h+1}) - \varphi_i(\boxed{h}) \right) \right)$. This expression is always zero, as needed, since $\varepsilon_i(\boxed{h+1}) = \varphi_i(\boxed{h}) \in \{0, 1\}$ in all of our standard crystals.

No full subcrystal of $\mathbb{B}^{\otimes k}$ has highest weight ϖ_k when ϖ_k is one of the spin fundamental weights in type B_n or D_n because all weights for $\mathbb{B}^{\otimes k}$ are in $\mathbb{Z}^n$, which does not contain either of the spin weights. □

To complete our construction of fundamental crystals for all fundamental weights, we need to identify "spin" crystals corresponding to ϖ_n in type B_n and to ϖ_{n-1} and ϖ_n in type D_n. We will provide such crystals using *minuscule weights*, which are defined as follows.

A weight $\lambda \in \Lambda$ is *minuscule* if $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\alpha \in \Phi$, where $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$.

Here is the classification of the minuscule fundamental weights in the irreducible Cartan types:

- In type A_n, all fundamental weights are minuscule.
- In type B_n, only the spin fundamental weight $\varpi_n = \frac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n)$ is minuscule.
- In type C_n, only $\varpi_1 = \mathbf{e}_1$ is minuscule.
- In type D_n, both $\varpi_1 = \mathbf{e}_1$ and the spin fundamental weights $\frac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n-1} \pm \mathbf{e}_n)$ are minuscule.
- In type E_6, there are two minuscule fundamental weights (ϖ_1, ϖ_6 in Bump and Schilling's notation).
- In type E_7, there is just one minuscule fundamental weight (ϖ_7 in Bump and Schilling's notation).
- In types F_4, G_2, and E_8, there are no minuscule fundamental weights.

Suppose $\lambda \in \Lambda^+$ is a minuscule dominant weight. We define an associated crystal $\mathcal{M}_\lambda$ as follows.

Recall that $W = \langle s_i : i \in I \rangle$ where $s_i = r_{\alpha_i} : x \mapsto x - \langle x, \alpha_i^\vee \rangle \alpha_i$.

As a set, $\mathcal{M}_\lambda$ consists of elements v_μ indexed by the weights $\mu \in \{w \cdot \lambda : w \in W\}$ in the W-orbit of λ.

The crystal operators $f_i, e_i : \mathcal{M}_\lambda \to \mathcal{M}_\lambda \sqcup \{0\}$ for $i \in I$ are given by the formulas

$$f_i(v_\mu) = \begin{cases} v_{\mu-\alpha_i} & \text{if } \langle \mu, \alpha_i^\vee \rangle = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad e_i(v_\mu) = \begin{cases} v_{\mu+\alpha_i} & \text{if } \langle \mu, \alpha_i^\vee \rangle = -1 \\ 0 & \text{otherwise.} \end{cases}$$

The weight map for $\mathcal{M}_\lambda$ is $\mathbf{wt}(v_\mu) = \mu$.

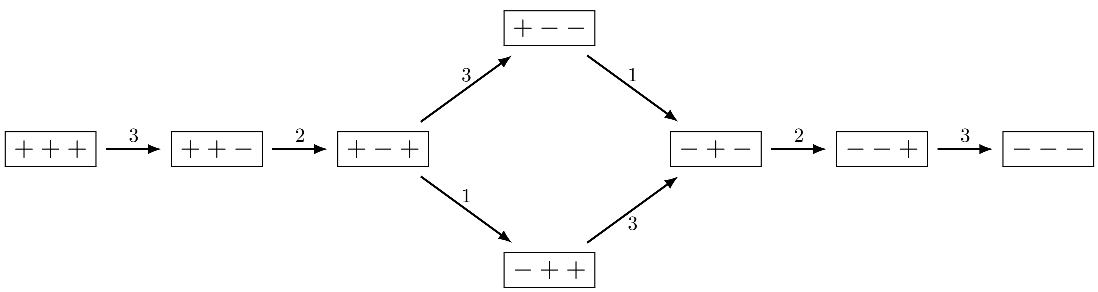
Example 3.3. Let's consider the B_n case of $\mathcal{M}_\lambda$ where $\lambda = \varpi_n = \frac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n)$.

The weights that can appear in the W-orbit of λ are $\mu = \frac{1}{2}\sum_{i=1}^n \epsilon_i \mathbf{e}_i$ where $\epsilon_i = \pm$.

Write this weight as the sequence of signs $\epsilon_1\epsilon_2\cdots\epsilon_n$ and the element $v_\mu \in \mathcal{M}_\lambda$ as $\boxed{\epsilon_1\epsilon_2\cdots\epsilon_n}$.

For example, $\boxed{++-}$ denotes v_μ for $\mu = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3)$.

The crystal graph of this example for $n = 3$ is shown below:



Proposition 3.4. Continue to assume $\lambda \in \Lambda^+$ is minuscule. Then there is a unique seminormal crystal structure on $\mathcal{M}_\lambda$ for the crystal operators and weight map just given. In this structure, v_λ is the unique highest weight element.

We omit the proof of this proposition, which is a routine exercise

Proposition 3.5. Continue to assume $\lambda \in \Lambda^+$ is minuscule. Let $\mathcal{B}$ be a seminormal crystal with a highest weight element u such that $\mathbf{wt}(u) = \lambda$. Assume $\mathcal{B}$ has the property that no two elements have the same weight. Then there is a unique crystal morphism $\psi : \mathcal{M}_\lambda \to \mathcal{B}$ with $\psi(v_\lambda) = u$.

Proof. By results in Lecture 5, these assumptions imply that $\mathcal{B}$ has a unique element b_μ of weight μ for every μ in the W-orbit of λ. The desired morphism is the map $\psi(v_\mu) = b_\mu$. □

Proposition 3.6. Suppose Φ is a simply-laced root system and $\lambda \in \Lambda^+$ is a minuscule dominant weight. Then $\mathcal{M}_\lambda$ is a Stembridge crystal.

Proof. One must show that every branched subcrystal of $\mathcal{M}_\lambda$ of types A_2 or $A_1 \times A_1$ is a Stembridge crystal. The connected components of such subcrystals satisfy the conditions of the previous proposition, so they are isomorphic to twists of minuscule crystals of types A_2 or $A_1 \times A_1$.

It is easy to work out the latter crystals explicitly: in type A_2 they are isomorphic to twists of $\mathrm{SSYT}_3(\lambda)$ for $\lambda = (1)$ and $\lambda = (1, 1)$, and something similar happens in type $A_1 \times A_1$. In either cases all crystals in sight are Stembridge crystals. □

We can now list explicit constructions of fundamental crystals:

Definition 3.7. For the spin fundamental weights ϖ_k in types B_n and D_n, we take $\mathcal{B}_{\varpi_k} = \mathcal{M}_{\varpi_k}$ to be the corresponding fundamental crystal. We refer to $\mathcal{B}_{\varpi_k}$ in these cases as a *spin crystal*.

For the other fundamental weights ϖ_k we take $\mathcal{B}_{\varpi_k}$ to be the full subcrystal described in Proposition 3.2.

The crystals $\mathcal{B}_{\varpi_k}$ will be called the *fundamental crystals* of classical type.

If ϖ_k is not a spin weight then $\mathcal{B}_{\varpi_k}$ is a Stembridge crystal or a virtual crystal (as full subcrystal of $\mathbb{B}^{\otimes k}$).

If ϖ_k is a spin weight for type D_n then $\mathcal{B}_{\varpi_k}$ is a Stembridge crystal by the previous proposition.

Finally, if ϖ_k is the spin weight for type B_n then $\mathcal{B}_{\varpi_k}$ is a virtual crystal:

Proposition 3.8. Let $\widehat{\mathcal{V}} = \mathcal{B}_{\varpi_n} \otimes \mathcal{B}_{\varpi_{n+1}}$ be the tensor product of the spin crystals for type D_{n+1}.

Let $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ be the subset generated by applying the virtual crystal operators

$$f_i = (\widehat{f}_i)^2 \text{ for } 1 \leq i < n \qquad \text{and} \qquad f_n = \widehat{f}_n \widehat{f}_{n+1}$$

to the highest weight element $v_{\varpi_n} \otimes v_{\varpi_{n+1}}$. Then $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ is a virtual crystal for the embedding $B_n \hookrightarrow D_{n+1}$ and this crystal is isomorphic to the spin crystal of type B_n.

Proof. One checks using the definition of minuscule crystals that $\varepsilon_i(b) = \frac{1}{2}\widehat{\varepsilon}_i(b)$ for $1 \leq i < n$ and $\varepsilon_n(b) = \widehat{\varepsilon}_n(b) = \widehat{\varepsilon}_{n+1}(b)$ for all $b \in V$, and similarly for φ_i. It follows that $\mathcal{V}$ is a virtual crystal.

One can also check that the map $\Psi : \Lambda^{B_n} \hookrightarrow \Lambda^{D_{n+1}}$ has $\Psi(\varpi_n^{B_n}) = \varpi_n^{D_{n+1}} + \varpi_{n+1}^{D_{n+1}}$.

Therefore $\mathcal{V}$ is a virtual crystal with the same highest highest weight as the spin crystal of type B_n. Using Proposition 3.5, one can now deduce that $\mathcal{V}$ is isomorphic to this spin crystal. $\square$

Corollary 3.9. Each fundamental crystal $\mathcal{B}_{\varpi_k}$ has a unique highest weight element with weight ϖ_k and is either a Stembridge crystal (in types A_n and D_n) or a virtual crystal (in types B_n or C_n).

4 Adjoint crystals

Let Φ be any root system with weight lattice Λ and simple roots $\{\alpha_i : i \in I\}$.

We define an *adjoint crystal* that is an analogue of the adjoint representation of the corresponding Lie algebra. This crystal $\mathcal{B}_{\mathsf{adjoint}}$ consists as a set of formal elements v_α for each root $\alpha \in \Phi$, along with an element $\tilde{v}_i$ for each $i \in I$. Define $\mathbf{wt}(v_\alpha) = \alpha$ and $\mathbf{wt}(\tilde{v}_i) = 0$. The crystal operators f_i are given by

$$f_i(v_\alpha) = \begin{cases} v_{\alpha-\alpha_i} & \text{if } \alpha - \alpha_i \in \Phi \\ \tilde{v}_i & \text{if } \alpha = \alpha_i \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f_i(\tilde{v}_j) = \begin{cases} v_{-\alpha_i} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

The crystal operators e_i are given by

$$e_i(v_\alpha) = \begin{cases} v_{\alpha+\alpha_i} & \text{if } \alpha + \alpha_i \in \Phi \\ \tilde{v}_i & \text{if } \alpha = -\alpha_i \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad e_i(\tilde{v}_j) = \begin{cases} v_{\alpha_i} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

The string lengths ε_i and φ_i are given in terms of the crystal operators by the usual seminormal formulas.

Proposition 4.1. The set $\mathcal{B}_{\mathsf{adjoint}}$ is a seminormal crystal with respect to the operators just given.

Lemma 4.2. Assume Φ is simply-laced and let $\alpha, \beta \in \Phi$

If $\alpha + k\beta \in \Phi$ for an integer $k > 1$, then $k = 2$ and $\alpha = -\beta$.

Proof. Consider the maximal string of roots

$$\alpha, \alpha + \beta, \alpha + 2\beta, \ldots, \alpha + k\beta \in \Phi.$$

Then if r_β is the reflection in the hyperplane orthogonal to $\beta^\vee$, we have $r_\beta(\alpha) = \alpha + k\beta$.

Therefore $k = \langle \alpha, \beta^\vee \rangle$. The intersection of Φ with the vector space spanned by α and β is a root system, and since α and β have the same length, this intersection is of type A_2 or $A_1 \times A_1$ or A_1 (when $\alpha = \pm\beta$). One checks that in these cases $\langle \alpha, \beta^\vee \rangle \leq 2$, with equality only when $\alpha = -\beta$. □

Proposition 4.3. Assume Φ is simple-laced. Then $\mathcal{B}_{\mathsf{adjoint}}$ is a Stembridge crystal.

Proof. By the lemma, the length k of the maximal string of roots

$$\alpha, \alpha + \beta, \alpha + 2\beta, \ldots, \alpha + k\beta \in \Phi$$

is bounded by 1, except when $\alpha = -\beta$ when $k = 2$. Thus $\varphi_i(v_\alpha) + \varepsilon_i(v_\alpha) \leq 1$ except when $\alpha = \pm\alpha_i$.

On the other hand $\varphi_i(v_\alpha) - \varepsilon_i(v_\alpha) = \langle \alpha, \alpha_i^\vee \rangle$ so we have

$$\varphi_i(v_\alpha) = \begin{cases} 2 & \text{if } \alpha = \alpha_i \\ 1 & \text{if } \langle \alpha, \alpha_i^\vee \rangle = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \varepsilon_i(v_\alpha) = \begin{cases} 2 & \text{if } \alpha = -\alpha_i \\ 1 & \text{if } \langle \alpha, \alpha_i^\vee \rangle = -1 \\ 0 & \text{otherwise} \end{cases}.$$

From these identities, the Stembridge axioms are easy to verify.

For more details, see Section 5.5 in Bump and Schilling's book. □

Question / exercise: in type A_{n-1}, what is $\mathcal{B}_{\mathsf{adjoint}}$ in terms of crystals of tableaux $\mathrm{SSYT}_n(\lambda)$?