1 Last time: properties of virtual crystals, specific constructions

For each embedding $X \hookrightarrow Y$ or type $C_n \hookrightarrow A_{2n-1}$ or $B_n \hookrightarrow D_{n+1}$ or $F_4 \hookrightarrow E_6$ or $G_2 \hookrightarrow D_4$ we have a notion of a *virtual crystal*. A virtual crystal $\mathcal{V}$ of type X is a subset of a Stembridge crystal $\widehat{\mathcal{V}}$ of type Y. This subset must be preserved by certain virtual crystal operators along with some other axioms.

We skip the precise definitions today, and just review three key properties proved last time:

Theorem 1.1. If $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ and $\mathcal{W} \subseteq \widehat{\mathcal{W}}$ are virtual crystals of type $X \hookrightarrow Y$ then so is $\mathcal{V} \otimes \mathcal{W} \subseteq \widehat{\mathcal{V}} \otimes \widehat{\mathcal{W}}$.

Theorem 1.2. Each connected virtual crystal has a unique highest weight element.

Theorem 1.3. If $\mathcal{V}, \mathcal{V}' \subseteq \widehat{\mathcal{V}}$ are connected virtual crystals with the same highest weights then $\mathcal{V} \cong \mathcal{V}'$.

Last time, we also discussed *fundamental crystals*, *minuscule crystals*, and *adjoint crystals*.

Let (Φ, Λ) be a Cartan type with simple roots $\{\alpha_i : i \in I\}$ and fundamental weights $\{\varpi_i : i \in I\}$.

A weight $\lambda \in \Lambda$ is *minuscule* if $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\alpha \in \Phi$, where $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$.

Suppose $\lambda \in \Lambda^+$ is minuscule. Recall that $W = \langle s_i : i \in I \rangle$ where $s_i = r_{\alpha_i} : x \mapsto x - \langle x, \alpha_i^\vee \rangle \alpha_i$.

Let $\mathcal{M}_\lambda$ be the set of elements v_μ indexed by the weights $\mu \in \{w \cdot \lambda : w \in W\}$ in the W-orbit of λ.

Let $\mathbf{wt}(v_\mu) = \mu$ and define crystal operators $f_i, e_i : \mathcal{M}_\lambda \to \mathcal{M}_\lambda \sqcup \{0\}$ for $i \in I$ by the formulas

$$f_i(v_\mu) = \begin{cases} v_{\mu-\alpha_i} & \text{if } \langle \mu, \alpha_i^\vee \rangle = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad e_i(v_\mu) = \begin{cases} v_{\mu+\alpha_i} & \text{if } \langle \mu, \alpha_i^\vee \rangle = -1 \\ 0 & \text{otherwise.} \end{cases}$$

There is a seminormal crystal structure on $\mathcal{M}_\lambda$ for these operators; call this a *minuscule crystal*.

Proposition 1.4. If Φ is a simply-laced root system then $\mathcal{M}_\lambda$ is a Stembridge crystal.

Assume $\Lambda = \mathbb{R}\text{-span}\{\varpi_i : i \in I\}$ is *simply-connected*. Let $\mathbb{B}$ be the standard crystal for (Φ, Λ).

Usually $\varpi_k = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_k$, but in types B_n and D_n are there are a few *spin fundamental weights*:

$$\varpi_n^{B_n} = \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_n), \quad \varpi_{n-1}^{D_n} = \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n-1} - \mathbf{e}_n), \quad \varpi_n^{D_n} = \tfrac{1}{2}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n-1} + \mathbf{e}_n).$$

Last time, we introduced *fundamental crystals* $\mathcal{B}_{\varpi_k}$ in all classical types for each fundamental weight ϖ_k:

- For the spin fundamental weights ϖ_k in types B_n and D_n, we defined $\mathcal{B}_{\varpi_k} = \mathcal{M}_{\varpi_k}$.
- Otherwise we defined $\mathcal{B}_{\varpi_k}$ to be the full subcrystal generated by $\boxed{k} \otimes \cdots \otimes \boxed{2} \otimes \boxed{1} \in \mathbb{B}^{\otimes k}$.

Proposition 1.5. Each fundamental crystal $\mathcal{B}_{\varpi_k}$ has a unique highest weight element with weight ϖ_k and is either a Stembridge crystal (in types A_n and D_n) or a virtual crystal (in types B_n or C_n).

Finally, we defined a seminormal crystal $\mathcal{B}_{\mathsf{adjoint}}$ consisting as a set of formal elements v_α for each $\alpha \in \Phi$, along with an element $\tilde{v}_i$ for each $i \in I$. The weight map has $\mathbf{wt}(v_\alpha) = \alpha$ and $\mathbf{wt}(\tilde{v}_i) = 0$, and we set

$$f_i(v_\alpha) = \begin{cases} v_{\alpha-\alpha_i} & \text{if } \alpha - \alpha_i \in \Phi \\ \tilde{v}_i & \text{if } \alpha = \alpha_i \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f_i(\tilde{v}_j) = \begin{cases} v_{-\alpha_i} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

with similar formulas for the e_i operators. (The f_i's determine the e_i's and vice versa in any crystal.)

Proposition 1.6. If Φ is a simple-laced root system then $\mathcal{B}_{\mathsf{adjoint}}$ is a Stembridge crystal.

2 Fundamental crystals in exceptional types

Besides the irreducible root systems of classical types A_n, B_n, C_n, and D_n, there are five exceptional roots systems of types E_6, E_7, E_8, F_4, and G_2. Let Φ be one of these exceptional root systems.

Let Λ be a weight lattice for this root system.

Assume Λ is simply-connected, i.e., spanned by the fundamental weights $\{\varpi_i : i \in I\}$.

One can likewise define fundamental crystals $\mathcal{B}_{\varpi_k}$ with unique highest weight ϖ_k.

The relevant constructions are summarized as follows:

- In type E_6, there are six fundamental weights ϖ_k, indexed by $k \in \{1,2,3,4,5,6\}$. For two of these (indices $k = 1$ and $k = 6$ in Bump and Schilling's notation), one realizes $\mathcal{B}_{\varpi_k} = \mathcal{M}_{\varpi_k}$ as a minuscule crystal. The remaining fundamental crystals $\mathcal{B}_{\varpi_k}$ are then defined as certain full subcrystals

$$\mathcal{B}_{\varpi_2} \subset \mathcal{B}_{\varpi_1} \otimes \mathcal{B}_{\varpi_6}, \quad \mathcal{B}_{\varpi_3} \subset \mathcal{B}_{\varpi_1} \otimes \mathcal{B}_{\varpi_1}, \quad \mathcal{B}_{\varpi_4} \subset \mathcal{B}_{\varpi_6} \otimes \mathcal{B}_{\varpi_6}, \quad \text{and} \quad \mathcal{B}_{\varpi_5} \subset \mathcal{B}_{\varpi_1} \otimes \mathcal{B}_{\varpi_6} \otimes \mathcal{B}_{\varpi_1}.$$

- In type E_7, there are seven fundamental weights ϖ_k, indexed by $k \in \{1,2,3,4,5,6,7\}$. For the last of these (in Bump and Schilling's notation), one sets $\mathcal{B}_{\varpi_7} = \mathcal{M}_{\varpi_7}$. The remaining fundamental crystals $\mathcal{B}_{\varpi_k}$ are then defined as certain full subcrystals

$$\mathcal{B}_{\varpi_1}, \mathcal{B}_{\varpi_6} \subset (\mathcal{B}_{\varpi_7})^{\otimes 2}, \quad \mathcal{B}_{\varpi_2}, \mathcal{B}_{\varpi_5} \subset (\mathcal{B}_{\varpi_7})^{\otimes 3}, \quad \text{and} \quad \mathcal{B}_{\varpi_3}, \mathcal{B}_{\varpi_4} \subset (\mathcal{B}_{\varpi_7})^{\otimes 4}.$$

- In type E_8, there are eight fundamental weights ϖ_k, indexed by $k \in \{1,2,3,4,5,6,7,8\}$. For the last of these (in Bump and Schilling's notation), one realizes $\mathcal{B}_{\varpi_8} = \mathcal{B}_{\mathsf{adjoint}}$ as the adjoint crystal for the corresponding root system. The remaining fundamental crystals $\mathcal{B}_{\varpi_k}$ are then defined as certain full subcrystals of tensor powers $(\mathcal{B}_{\mathsf{adjoint}})^{\otimes q}$ for $q = 2,3,4,5$.
- In type F_4, there are four fundamental weights $\varpi_1, \varpi_2, \varpi_3, \varpi_4$. The last fundamental crystal $\mathcal{B}_{\varpi_4}$ is defined as a certain virtual crystal inside $\mathcal{B}_{\varpi_1} \otimes \mathcal{B}_{\varpi_6} = \mathcal{M}_{\varpi_1} \otimes \mathcal{M}_{\varpi_6}$ for the embedding $F_4 \hookrightarrow E_6$. The remaining fundamental crystals are given as full subcrystals of tensor powers of $\mathcal{B}_{\varpi_4}$.
- In type G_2, there are just two fundamental weights ϖ_1 and ϖ_2. The fundamental crystal $\mathcal{B}_{\varpi_1}$ is defined as a certain virtual crystal inside $\mathcal{B}_{\varpi_1} \otimes \mathcal{B}_{\varpi_3} \otimes \mathcal{B}_{\varpi_4}$ for the embedding $G_2 \hookrightarrow D_4$. The other fundamental crystal is a full subcrystal of $\mathcal{B}_{\varpi_1} \otimes \mathcal{B}_{\varpi_1}$.

More complete details about these constructions appear in Section 5.6 of Bump and Schilling's book.

The key point is that for each fundamental weight ϖ_k in each type, we have a crystal $\mathcal{B}_{\varpi_k}$ such that:

- $\mathcal{B}_{\varpi_k}$ is connected with a unique highest weight element whose weight is $\varpi_k \in \Lambda$.
- $\mathcal{B}_{\varpi_k}$ is a Stembridge crystal in simply-laced types A, D, E.
- $\mathcal{B}_{\varpi_k}$ is a virtual crystal in the remaining types B, C, F, G.

Important point: our seemingly *ad hoc* definitions of these crystals are actually canonical, because the isomorphism class of a connected Stembridge or virtual crystal is uniquely determined by its highest weight. Thus, the properties just listed uniquely determine the crystals $\mathcal{B}_{\varpi_k}$ up to isomorphism.

3 Normal crystals for general Cartan types

Recall that *twisting* a crystal $\mathcal{B}$ of type (Φ, Λ) refers to shifting the values of the weight map by some fixed element of Λ that is orthogonal to every root in Φ.

The axioms of a Stembridge crystal are preserved under twisting. However, our definition of a virtual crystal required (Φ, Λ) to be semisimple (meaning that the ambient space V is spanned by $\Phi \subset \Lambda$).

To remove this requirement, we define a crystal of Cartan type B_n, C_n, G_2, or F_4 (with any weight lattice) to be *virtualizable* if its isomorphic to a twist of a virtual crystal.

Explicitly, a crystal $\mathcal{B}$ of one of these types is virtualizable if there exists some weight $\eta \in \Lambda$ that is orthogonal to all simple roots such that $\mathbf{wt}(b) + \eta \in \mathbb{R}\Phi$ for all $b \in \mathcal{B}$.

Definition 3.1. Suppose $\mathcal{B}$ is a crystal of Cartan type (Φ, Λ). If the root system Φ is irreducible, then we define $\mathcal{B}$ to be *normal* if it is either a Stembridge crystal or a virtualizable crystal.

If the root system Φ is reducible, then (Φ, Λ) decomposes as a product of a finite number of irreducible Cartan types, and we define $\mathcal{B}$ to be *normal* if it is a direct product of normal crystals for each irreducible factor.

The motivation for this definition comes from results of Kashiwara in the 1990s, which show that normal crystals are isomorphic to the "crystal bases" of representations of quantized enveloping algebras.

Later in the course, we will see that the characters of connected normal crystals are exactly the characters of the corresponding irreducible Lie group representation with the same highest weight.

We confine our attention today to the nice algebraic properties of the class of normal crystals, which one should expect from these connections to representation theory.

Theorem 3.2. Fix a Cartan type (Φ, Λ).

The class of normal crystals for this type is closed under tensor products and twisting.

Every normal crystal is seminormal. Every full subcrystal of a normal crystal is normal.

Every connected normal crystal has a unique highest weight element.

(The weight of a highest weight element is always in the set of dominant weights Λ^+.)

For any $\lambda \in \Lambda^+$, there is a unique isomorphism class of connected normal crystals with highest weight λ.

Proof. It is enough to consider the case when (Φ, Λ) is an irreducible Cartan type.

The only part of the theorem that has not been shown in earlier lectures is the last claim, that for any dominant weight $\lambda \in \Lambda^+$ there exists a connected normal crystal with unique highest weight λ.

If $\lambda = \varpi_i$ is a fundamental weight then the fundamental crystal $\mathcal{B}_{\varpi_i}$ is a crystal with the desired property.

Suppose $\lambda \in \Lambda^+$ is an arbitrary dominant weight and let $\lambda' = \sum_{i \in I} c_i \varpi_i$ where $c_i = \langle \lambda, \alpha_i^\vee \rangle$.

If u_{ϖ_i} is the highest weight element of $\mathcal{B}_{\varpi_i}$ then $\bigotimes_{i\in I} u_{\varpi_i}^{\otimes c_i}$ is a highest weight element in $\bigotimes_{i \in I} \mathcal{B}_{\varpi_i}^{\otimes c_i}$.

This element generates a full subcrystal $\mathcal{B}_{\lambda'}$ with unique highest weight λ'.

As a subcrystal of a tensor product of normal crystals, $\mathcal{B}_{\lambda'}$ is normal.

We can twist $\mathcal{B}_{\lambda'}$ by $\eta = \lambda - \lambda'$ since η is orthogonal to all simple (co)roots.

The twisted crystal $\mathcal{B}_\lambda$ is then connected and normal with unique highest weight λ, as needed. □

As with Stembridge crystals, we can detect a normal crystal from its subcrystals of rank two:

Theorem 3.3. Let $\mathcal{B}$ be a finite crystal for a Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$.

Suppose that for all distinct indices $i, j \in I$, the branched subcrystal $\mathcal{B}_J$ for $J = \{i, j\}$ is normal as a crystal for the rank two root system Φ_J generated by α_i and α_j. Then $\mathcal{B}$ is normal.

Proof. Here is a sketch of the proof from Bump and Schilling's book (Theorem 5.21).

Every normal crystal is seminormal, so our hypotheses imply that $\mathcal{B}$ is seminormal.

Let u_λ be a highest weight element of $\mathcal{B}$ with weight λ.

Let $\mathcal{B}'$ be a connected normal crystal with highest weight λ. Write u'_λ for its highest weight element.

Consider pairs of subsets (Ω, Ω') with $u_\lambda \in \Omega \subseteq \mathcal{B}$ and $u'_\lambda \in \Omega' \subseteq \mathcal{B}'$ equipped with a bijection $\Omega \to \Omega'$, written $x \mapsto x'$, such that:

- We always have $\mathbf{wt}(x) = \mathbf{wt}(x')$.
- We have $u_\lambda \mapsto u'_\lambda$.
- If $x \in \Omega$ and $e_i(x) \neq 0$ then $e_i(x) \in \Omega$ and $(e_i(x))' = e_i(x') \in \Omega'$.

There is at least one such pair: take $\Omega = \{u_\lambda\}$ and $\Omega' = \{u'_\lambda\}$.

So we can assume that (Ω, Ω') is chosen such that Ω is as large as possible. We will argue that $\Omega' = \mathcal{B}'$ in order to conclude that the full subcrystal of $\mathcal{B}$ containing u_λ is isomorphic to $\mathcal{B}'$ and therefore normal.

Suppose $\Omega' \subsetneq \mathcal{B}'$ and $w' \in \mathcal{B}' \setminus \Omega'$ is maximal under the order generated by $w' \prec e_i(w')$.

We will show that this contradicts our assumption of maximality.

Since $w' \neq u'_\lambda$, there is an index i with $e_i(w') \neq 0$, and we must have $y' := e_i(w') \in \Omega'$.

Let $y \in \Omega$ be the preimage of y'. It is straightforward to check that $\varepsilon_i(y) = \varepsilon_i(y')$, and since our crystals are seminormal and the bijection $\Omega \to \Omega'$ is weight-preserving, it follows that $\varphi_i(y) = \varphi_i(y')$.

Since $f_i(y') = w' \neq 0$, we have $\varphi_i(y) = \varphi_i(y') > 0$, so $y = e_i(w)$ for some $w \in \mathcal{B} \setminus \Omega$.

The next step in the proof is to show that our construction of w does not depend on the choice of $i \in I$. This is not hard to show using the assumption that $\mathcal{B}_J$ is normal for any subset $J = \{i, j\} \subset I$.

Since w does not depend on the choice of $i \in I$, it follows that we can extend the map $\Omega \to \Omega'$ to a bijection $\Omega \sqcup \{w\} \to \Omega' \sqcup \{w'\}$ retaining the desired properties. As this contradicts the maximality of Ω, we must have $\Omega' = \mathcal{B}'$ as claimed. □

To state our last general property of normal crystals, we need to clarify our definition of virtual crystals for reducible Cartan types. Consider the situation in which $\mathcal{B}$ is a crystal of simply-laced Cartan type X and let $Y = X \times X$. Then Y is also a simply-laced Cartan type.

Let I^X be the index set for X. The index set for $Y = I_1 \sqcup I_2$ is the disjoint union of two copies I_1 and I_2 of I^X. If $i \in I^X$ then we denote by i_1 and i_2 the corresponding elements of I_1 and I_2.

Define $\sigma : I^X \to \{$ subsets of I^Y $\}$ by $\sigma(i) = \{i_1, i_2\}$ and set $\gamma_i = 1$ for all $i \in I^X$. Finally let $\widehat{\mathcal{B}} = \mathcal{B} \times \mathcal{B}$; this is a crystal of type Y. We then consider $\mathcal{V} = \{(u, u) : u \in \mathcal{B}\}$ to be a virtual crystal inside $\widehat{\mathcal{V}}$.

The setup here is identical to what we did for virtual crystals in Lecture 7, except now X is simply-laced.

With these conventions in place, the following is straightforward:

Theorem 3.4. The class of normal crystals is preserved by Levi branching.

4 Similarity of crystals

Fix an arbitrary Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$.

For each $\lambda \in \Lambda^+$, let $\mathcal{B}_\lambda$ be a connected normal crystal of type (Φ, Λ) with unique highest weight λ.

Proposition 4.1. Suppose $\lambda, \mu \in \Lambda^+$. Then there exists a crystal embedding $\mathcal{B}_{\lambda+\mu} \hookrightarrow \mathcal{B}_\lambda \otimes \mathcal{B}_\mu$.

Proof. Let u_λ and u_μ be the highest weight elements in $\mathcal{B}_\lambda$ and $\mathcal{B}_\mu$ respectively.

Then $u_\lambda \otimes u_\mu$ is a highest weight element in $\mathcal{B}_\lambda \otimes \mathcal{B}_\mu$ since $\varepsilon_i(u_\lambda \otimes u_\mu) = 0$ for $i \in I$.

The connected component containing this highest weight element is normal.

This full subcrystal is therefore isomorphic to $\mathcal{B}_{\lambda+\mu}$ since $\lambda + \mu = \mathbf{wt}(u_\lambda \otimes u_\mu)$. □

Fix a positive integer n and a dominant $\lambda \in \Lambda^+$.

Suppose $S : \mathcal{B}_\lambda \to \mathcal{B}_{n\lambda}$ is a map such that $\mathbf{wt}(S(v)) = n\mathbf{wt}(v)$ and

$$\varphi_i(S(v)) = n\varphi_i(v), \qquad \varepsilon_i(S(v)) = n\varepsilon_i(v), \qquad S(e_i(v)) = e_i^n(S(v)), \qquad S(f_i(v)) = f_i^n(S(v)) \tag{4.1}$$

for all $v \in \mathcal{B}_\lambda$ and $i \in I$. One calls S a *similarity map of degree n*.

If a similarity map $\mathcal{B}_\lambda \to \mathcal{B}_{n\lambda}$ exists then it is unique since it must preserve highest weight elements.

The main application of similarity maps is to show that two of the original approaches to crystals from representation theory coincide. Namely, crystals obtained by the so-called Littelmann path method are the same as those derived from crystal bases of representations. To reach this conclusion one needs to show that similarity maps exist for all normal crystals and all Cartan types.

Here are some preliminary results in this direction:

Proposition 4.2. Suppose that $S_\lambda : \mathcal{B}_\lambda \to \mathcal{B}_{n\lambda}$ and $S_\mu : \mathcal{B}_\mu \to \mathcal{B}_{n\mu}$ are similarity maps of degree n.

Then there exists a similarity map $\mathcal{B}_{\lambda+\mu} \to \mathcal{B}_{n(\lambda+\mu)}$.

Proof. We can identify the crystal $\mathcal{B}_{\lambda+\mu}$ with a full subcrystal of $\mathcal{B}_\lambda \otimes \mathcal{B}_\mu$.

We argue that the desired similarity map $S : \mathcal{B}_{\lambda+\mu} \to \mathcal{B}_{n(\lambda+\mu)}$ is defined by

$$S(u \otimes v) = S_\lambda(u) \otimes S_\mu(v).$$

It is clear that $\mathbf{wt}(S(u \otimes v)) = n\mathbf{wt}(u) + n\mathbf{wt}(v) = n\mathbf{wt}(u \otimes v)$. Next, observe that

$$\varphi_i(S(u \otimes v)) = \varphi_i(S_\lambda(u) \otimes S_\mu(v)) = \max\{\varphi_i(S_\lambda(u)), \varphi_i(S_\lambda(u)) + \varphi_i(S_\mu(v)) - \varepsilon_i(S_\lambda(u))\}$$

by the definition of string lengths for tensor products of seminormal crystals.

Substituting the identities in (4.1) shows that $\varphi_i(S(u \otimes v)) = n\varphi_i(u \otimes v)$ as needed.

It follows similarly that $\varepsilon_i(S(u \otimes v)) = n\varepsilon_i(u \otimes v)$. Finally, we have

$$f_i^n(S_\lambda(u) \otimes S_\mu(v)) = \begin{cases} f_i^n(S_\lambda(u)) \otimes S_\mu(v) & \text{if } \varphi_i(S_\mu(v)) \leq \varepsilon_i(S_\lambda(u)) \\ S_\lambda(u) \otimes f_i^n(S_\mu(v)) & \text{if } \varphi_i(S_\mu(v)) \geq \varepsilon_i(S_\lambda(u)) + n. \end{cases}$$

We cannot have $\varepsilon_i(S_\lambda(u)) < \varphi_i(S_\mu(v)) < \varepsilon_i(S_\lambda(u)) + n$ since all string lengths are multiples of n.

Since $f_i^n(S_\lambda(u)) \otimes S_\mu(v) = S(f_i(u) \otimes v)$ and $S_\lambda(u) \otimes f_i^n(S_\mu(v)) = S(u \otimes f_i(v))$, it follows that

$$S(f_i(u \otimes v)) = f_i^n(S(u \otimes v))$$

as needed. The required identity for e_i follows similarly. □

Proposition 4.3. If (Φ, Λ) is of type A_n, then any $\mathcal{B}_\lambda$ has a similarity map for any degree $m > 0$.

Proof. Write $\mathbb{B} = \mathbb{B}_{n+1}$ for the standard crystal of type A_n (i.e., $\mathrm{GL}(n+1)$).

Then $\mathcal{B}_\lambda$ is isomorphic to a full subcrystal of $\mathbb{B}^{\otimes k}$ for some k.

By the previous result, it is enough to show that $\mathbb{B} = \mathcal{B}_{\varpi_1}$ has a similarity map to $\mathcal{B}_{m\varpi_1}$.

It is easy and instructive to check that the map $\boxed{i} \mapsto \boxed{i\,|\,i\,|\cdots|\,i} \in \mathrm{SSYT}_{n+1}((m))$ is such a map. □