1 Review from last time

1.1 Normal crystals for general Cartan types

Twisting a crystal means shifting the values of the weight map by a fixed weight orthogonal to all roots.

In simply-laced types (A, D, E) we have a class of Stembridge crystals, which is preserved by twisting.

In non-simply laced types (B, C, F, G) we define a *virtualizable crystal* to be a twist of a virtual crystal.

A Cartan type (Φ, Λ) is *irreducible* if the root system Φ is irreducible.

Definition 1.1. A crystal for an irreducible Cartan type if *normal* if it is either a Stembridge crystal or a virtualizable crystal. A crystal for a reducible Cartan type is *normal* if it is a direct product of normal crystals for irreducible Cartan types.

Normal crystals are isomorphic to the "crystal bases" of representations of quantized enveloping algebras.

Here is a summary of the main algebraic properties of normal crystals for a given Cartan type:

- Tensor products and twists of normal crystals are also normal.
- Every normal crystal is seminormal. Every full subcrystal of a normal crystal is normal.
- Every connected normal crystal has a unique highest weight element.
- Suppose $\mathcal{B}$ and $\mathcal{C}$ are two connected normal crystals with the same highest weight.

  Then $\mathcal{B} \cong \mathcal{C}$. More strongly, there is a *unique* crystal isomorphism $\mathcal{B} \xrightarrow{\sim} \mathcal{C}$.
- There exists a connected normal crystal with any given dominant weight as its highest weight.
- Any Levi branched subcrystal of a normal crystal is normal.
- Finally, if $\mathcal{B}$ is finite and the subcrystal $\mathcal{B}_J$ is normal for all pairs $J = \{i, j\}$ then $\mathcal{B}$ is normal.

1.2 Similarity of crystals

Fix a Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$. Let n be a positive integer.

Fix $\lambda \in \Lambda^+$. Let $\mathcal{B}_\lambda$ be a connected normal crystal of type (Φ, Λ) with unique highest weight λ.

A map $S : \mathcal{B}_\lambda \to \mathcal{B}_{n\lambda}$ is called a *degree n similarity* if $\mathbf{wt}(S(v)) = n\mathbf{wt}(v)$ and

$$\varphi_i(S(v)) = n\varphi_i(v), \qquad \varepsilon_i(S(v)) = n\varepsilon_i(v), \qquad S(e_i(v)) = e_i^n(S(v)), \qquad S(f_i(v)) = f_i^n(S(v)) \tag{1.1}$$

for all $v \in \mathcal{B}_\lambda$ and $i \in I$. Some facts from last time:

- If a similarity map $\mathcal{B}_\lambda \to \mathcal{B}_{n\lambda}$ exists then it is unique.
- If there are similarity maps $\mathcal{B}_\lambda \to \mathcal{B}_{n\lambda}$ and $\mathcal{B}_\mu \to \mathcal{B}_{n\mu}$ then there is a similarity map $\mathcal{B}_{\lambda+\mu} \to \mathcal{B}_{n(\lambda+\mu)}$.
- If (Φ, Λ) is of type A, then each $\mathcal{B}_\lambda$ has a similarity map for any degree $n > 0$.

2 The plactic monoid

Today's lecture, combined with Homework 1, will cover most of Chapter 8 in Bump and Schilling's book.

The *plactic monoid* is a multiplicative structure on semistandard tableaux first studied by Lascoux and Schützenberger in the 1980s. One encounters this object in a natural way through $\mathrm{GL}(n)$ crystals.

We start with a definition of *plactic equivalence* that applies to any Cartan type.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be normal crystals of the same Cartan type.

Suppose $x_i \in \mathcal{C}_i$ and $\mathcal{C}'_i \subseteq \mathcal{C}_i$ is the connected component containing x_i for each $i = 1, 2$.

Definition 2.1. If $\mathcal{C}'_1$ is isomorphic to $\mathcal{C}'_2$, and if the unique isomorphism $\mathcal{C}'_1 \xrightarrow{\sim} \mathcal{C}'_2$ maps $x_1 \mapsto x_2$, then we write $x_1 \equiv x_2$ and say that the two elements are *plactically equivalent*.

It is easy to check that this definition of $\equiv$ gives an equivalence relation.

Observation 2.2. If $x_1, y_1 \in \mathcal{C}_1$ and $x_2, y_2 \in \mathcal{C}_2$ are such that $x_1 \equiv x_2$ and $y_1 \equiv y_2$, then $x_1 \otimes y_1 \equiv x_2 \otimes y_2$.

Proof. This is clear from the definition of $\equiv$ on noting that if f and g are crystal isomorphisms then the map $x \otimes y \mapsto f(x) \otimes g(y)$ is an isomorphism between the corresponding tensor products. □

Now suppose $\mathcal{B}$ is a fixed normal crystal.

Define $\mathsf{Plactic}(\mathcal{B})$ to be the set of equivalence classes in

$$\{\emptyset\} \sqcup \mathcal{B} \sqcup (\mathcal{B} \otimes \mathcal{B}) \sqcup (\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}) \sqcup \cdots$$

under $\equiv$, where $\emptyset$ denotes a distinguished "empty tensor" that is equivalent only to itself.

Observation 2.3. The tensor product $\otimes$ induces a monoid structure on the (infinite) set $\mathsf{Plactic}(\mathcal{B})$.

Explicitly, if we write $[w]$ for the equivalence class of w, then the monoid structure on $\mathsf{Plactic}(\mathcal{B})$ has

$$[a_1 \otimes a_2 \otimes \cdots \otimes a_p][b_1 \otimes b_2 \otimes \cdots \otimes b_q] = [a_1 \otimes a_2 \otimes \cdots \otimes a_p \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_q].$$

The identity element is $[\emptyset]$ where we define $\emptyset \otimes b = b \otimes \emptyset = b$.

For the rest of this section we fix a positive integer n and specialize to Cartan type $\mathrm{GL}(n)$.

Let $\mathbb{B} = \mathbb{B}_n$ denote the standard $\mathrm{GL}(n)$ crystal. The *(type A) plactic monoid* is $\mathsf{Plactic}(\mathbb{B})$.

As usual we write tensors $w_1 \otimes w_2 \otimes \cdots \otimes w_m \in \mathbb{B}^{\otimes m}$ where each $w_i \in \{1, 2, \ldots, n\}$ as words $w_1 w_2 \cdots w_m$.

Using Homework 1, we can give a more explicit description of plactic equivalence for $\mathsf{Plactic}(\mathbb{B})$.

Suppose $v = v_1 v_2 \cdots v_m$ and $w = w_1 w_2 \cdots w_m$ are words in $\mathbb{B}^{\otimes m}$ of the same length.

Recall from Homework 1 that we say v and w are connected by a *Knuth move* if w is obtained from v by applying one of the following transformations to three consecutive letters, assuming $a < b < c$:

$$cab \leftrightarrow acb, \qquad bac \leftrightarrow bca, \qquad aba \leftrightarrow baa, \qquad bba \leftrightarrow bab$$

This happens, for example, if $v = 433574$ and $w = 343574$ or $w = 433547$.

Knuth equivalence is the equivalence relation on words that has $v \overset{\mathsf{K}}{\sim} w$ if and only if v and w are connected by a sequence of Knuth moves. For example, $43534 \overset{\mathsf{K}}{\sim} 43354 \overset{\mathsf{K}}{\sim} 34354 \overset{\mathsf{K}}{\sim} 34534$.

Recall the definition of the RSK correspondence $w \mapsto (P_{\mathsf{RSK}}(w), Q_{\mathsf{RSK}}(w))$ from Lecture 3.

On Homework 1, you proved the following for any words v and w:

Theorem 2.4. One always has $w \overset{\mathsf{K}}{\sim} \mathfrak{row}(P_{\mathsf{RSK}}(w))$, and $P_{\mathsf{RSK}}(v) = P_{\mathsf{RSK}}(w)$ holds if and only if $v \overset{\mathsf{K}}{\sim} w$.

Example 2.5. For example, we have

$$\boxed{4} \rightsquigarrow \begin{array}{l} 3 \\ 4 \end{array} \rightsquigarrow \begin{array}{ll} 3 & 5 \\ 4 & \end{array} \rightsquigarrow \begin{array}{ll} 3 & 3 \\ 4 & 5 \end{array} \rightsquigarrow \begin{array}{lll} 3 & 3 & 4 \\ 4 & 5 & \end{array} = P_{\mathsf{RSK}}(43534)$$

$$\boxed{3} \rightsquigarrow \begin{array}{ll} 3 & 4 \end{array} \rightsquigarrow \begin{array}{lll} 3 & 4 & 5 \end{array} \rightsquigarrow \begin{array}{lll} 3 & 3 & 5 \\ 4 & & \end{array} \rightsquigarrow \begin{array}{lll} 3 & 3 & 4 \\ 4 & 5 & \end{array} = P_{\mathsf{RSK}}(34534).$$

On the other hand, combining results from Lecture 3 and Homework 1 gives the following theorem:

Theorem 2.6. Let m be a positive integer. Then $\mathbb{B}^{\otimes m}$ is a disjoint union of full subcrystals isomorphic to crystals of semistandard tableaux $\mathrm{SSYT}_n(\lambda)$ for partitions λ of m with at most n parts.

More concretely, the full subcrystal containing $w \in \mathbb{B}^{\otimes m}$ is isomorphic to $\mathrm{SSYT}_n(\lambda)$ where λ is the shape of $P_{\mathsf{RSK}}(w)$, and the map $x \mapsto P_{\mathsf{RSK}}(x)$ is the unique isomorphism from this subcrystal to $\mathrm{SSYT}_n(\lambda)$.

Also, two words $v, w \in \mathbb{B}^{\otimes m}$ belong to the same full subcrystal if and only if $Q_{\mathsf{RSK}}(v) = Q_{\mathsf{RSK}}(w)$.

Corollary 2.7. Plactic equivalence $\equiv$ for $\mathsf{Plactic}(\mathbb{B})$ is the same thing as Knuth equivalence $\overset{\mathsf{K}}{\sim}$.

Proof. The theorem shows that $w \in \mathbb{B}^{\otimes m}$ is plactically equivalent to $P_{\mathsf{RSK}}(w) \in \mathrm{SSYT}_n(\lambda)$ where λ is the shape of $P_{\mathsf{RSK}}(w)$. It follows that if $v, w \in \mathbb{B}^{\otimes m}$ and $v \overset{\mathsf{K}}{\sim} w$ then $v \equiv P_{\mathsf{RSK}}(v) = P_{\mathsf{RSK}}(w) \equiv w$.

Conversely if $v \equiv w$ then $P_{\mathsf{RSK}}(v) \equiv P_{\mathsf{RSK}}(w)$ so $P_{\mathsf{RSK}}(v)$ and $P_{\mathsf{RSK}}(w)$ must have the same shape λ. Since the identity map is the unique automorphism of the connected normal crystal $\mathrm{SSYT}_n(\lambda)$, we conclude that if $v \equiv w$ then $P_{\mathsf{RSK}}(v) = P_{\mathsf{RSK}}(w)$ and $v \overset{\mathsf{K}}{\sim} w$. □

Thus, the elements of $\mathsf{Plactic}(\mathbb{B})$ are Knuth equivalence classes, which are evidently in bijection with semistandard tableaux. This allows us to transfer the monoid structure on $\mathsf{Plactic}(\mathbb{B})$ to tableaux:

Corollary 2.8. The set of SSYT_n all semistandard tableaux with entries in $\{1, 2, \dots, n\}$ has a unique monoid structure in which $U \circ V = P_{\mathsf{RSK}}(\mathfrak{row}(U)\mathfrak{row}(V))$. Here, the identity is the unique empty tableau.

The monoid algebra $\mathbb{Z}[\mathrm{SSYT}_n]$ associated to $(\mathrm{SSYT}_n, \circ)$ is sometimes called the *Poirier-Reutenaurer algebra*. It has a natural Hopf algebra structure.

The unique highest weight element in $\mathrm{SSYT}_n(\lambda)$ is the tableau whose entries in row i are all i.

Thus the highest weight elements in $\mathbb{B}^{\otimes m}$ are the words w for which $P_{\mathsf{RSK}}(w)$ is a tableau of this form.

One can characterize such words more directly.

Definition 2.9. A word $w = w_1 w_2 \cdots w_m$ is a *Yamanouchi word* (or a *reverse lattice word*) if for each positive integer i all of the final segments $w_{k+1}w_{k+2}\cdots w_m$ contain at least as many letters equal to i as $i+1$.

Note that this condition means that a Yamanouchi word must end in the letter 1.

Proposition 2.10. A word $w \in \mathbb{B}^{\otimes m}$ is a highest weight element if and only if it is a Yamanouchi word.

Proof. The general formula for the string length ε_i of a tensor product of m finite type crystals is

$$\varepsilon_i(x_m \otimes \cdots \otimes x_2 \otimes x_1) = \max_{j=1}^{m} \left(\sum_{h=1}^{j} \varepsilon_i(x_h) - \sum_{h=1}^{j-1} \varphi_i(x_h) \right).$$

Derive this by induction or see Section 2.3 of Bump and Schilling's book.

Applying this formula with $x_j = w_{m+1-j}$ gives

$$\varepsilon_i(w) = \max_{j=1}^{m} \left(\sum_{h=j}^{m} \varepsilon_i(w_h) - \sum_{h=j+1}^{m} \varphi_i(w_h) \right).$$

If the maximum is not zero then it is attained where $w_j = i+1$, in which case $\varphi_i(w_j) = 0$, so the formula is unchanged if we add this term to the second sum. But then the difference in the summations is counting exactly the difference between the number of $i+1$'s and i's in each final segment of w, so the condition that $\varepsilon_i(w) = 0$ for all i is equivalent to requiring that w be a Yamanouchi word. $\square$

3 Crystals of skew tableaux

A *skew shape* is an ordered pair of partitions (λ, μ) with $\mathsf{D}_\lambda \subseteq \mathsf{D}_\mu$, where as usual

$$\mathsf{D}_\lambda = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} : 1 \le j \le \lambda_i\}.$$

We write λ/μ in place of (λ,μ) and define $\mathsf{D}_{\lambda/\mu} = \mathsf{D}_\lambda \setminus \mathsf{D}_\mu$.

A *skew tableau* of shape λ/μ is a map $T : \mathsf{D}_{\lambda/\mu} \to \{1,2,3,\dots\}$. Such a map is *semistandard* if its rows are weakly increasing and its columns are strictly increasing. For example:

$$T = \begin{array}{|c|c|c|c|c|} \hline & & 1 & 2 & 2 \\ \hline & 1 & 2 & 4 & \\ \hline 1 & 2 & 3 & 5 & \\ \hline 2 & 4 & 4 & & \\ \hline \end{array}$$

is semistandard with shape $\lambda/\mu = (5,4,4,3)/(2,1)$.

The *reading word* $\mathfrak{row}(T)$ of a skew tableau is defined in the same way as for an ordinary tableau, by concatenating the rows starting with the last row. In our example, $\mathfrak{row}(T) = 2441235124122$.

Skew shapes and skew tableaux reduce to ordinary partitions and tableau on setting $\mu = \emptyset$.

Let $\mathrm{SSYT}_n(\lambda/\mu)$ denote the set of all semistandard skew tableaux of shape λ/μ with entries in $\{1,2,\dots,n\}$.

Theorem 3.1. Suppose λ/μ is a skew shape with $m = |\lambda| - |\mu|$.

The set of words $\mathfrak{row}(T) \in \mathbb{B}^{\otimes m}$ for $T \in \mathrm{SSYT}_n(\lambda/\mu)$ is then a (not necessarily full) subcrystal of $\mathbb{B}^{\otimes m}$.

Consequently, there is a unique $\mathrm{GL}(n)$ crystal structure on $\mathrm{SSYT}_n(\lambda/\mu)$ such that we have a morphism

$$\mathfrak{row} : \mathrm{SSYT}_n(\lambda/\mu) \to \mathbb{B}^{\otimes m}.$$

Proof. The result follows by essentially the same argument as in the $\mu = \emptyset$ case given in Lecture 2. $\square$

The *skew Schur polynomial* of shape λ/μ is $s_{\lambda/\mu}(x_1,x_2,\dots,x_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda/\mu)} x^{\mathbf{wt}(T)}$.

Corollary 3.2. The skew Schur polynomial $s_{\lambda/\mu}(x_1,x_2,\dots,x_n) = \mathrm{ch}(\mathrm{SSYT}_n(\lambda/\mu))$ is symmetric.

Since $\mathrm{SSYT}_n(\lambda/\mu)$ is a normal $\mathrm{GL}(n)$ crystal, it is isomorphic to a direct sum of $\mathrm{GL}(n)$ crystals of the form $\mathrm{SSYT}_n(\nu)$ where ν is a partition of m. Ignoring trailing zeros, the relevant partitions ν are the weights of the highest weight elements in $\mathrm{SSYT}_n(\lambda/\mu)$. To count these partitions, let $c^\lambda_{\mu\nu}$ be the number of semistandard skew tableau of shape λ/μ with weight ν whose reading words are Yamanouchi words.

Proposition 3.3. If n is sufficiently large then $\mathrm{SSYT}_n(\lambda/\mu) \cong \bigsqcup_\nu \mathrm{SSYT}_n(\nu)^{\otimes c^\lambda_{\mu\nu}}$.

Proof. The tableaux counted by $c^\lambda_{\mu\nu}$ are exactly the highest weight elements of weight ν.

Note that if $\mathfrak{row}(T)$ is Yamanouchi then T can only involve entries in $\{1, 2, \dots, k\}$ where k is the number of rows of T. If n is at least the number of rows of $\mathsf{D}_{\lambda/\mu}$ then the given isomorphism holds; otherwise some of the sets involved could be empty. □

Let $s_{\lambda/\mu} = \lim_{n\to\infty} s_{\lambda/\mu}(x_1, x_2, \dots, x_n)$, where the limit is in the sense of the formal power series (i.e., the limit exists if the coefficient of any fixed monomial is eventually constant.)

The symmetric power series $s_{\lambda/\mu}$ is the *skew Schur function* of shape λ/μ.

Corollary 3.4. It holds that $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$.

It turns out that the numbers $c^\lambda_{\mu\nu}$ are the same as we what called *Littlewood-Richardson coefficients* in Lectures 3 and 4, but proving this is slightly outside the scope of what we will accomplish today.

Example 3.5. Suppose $\lambda/\mu = (2, 2)/(1)$.

There is just one Yamanouchi word 121 that is the reading word of a semistandard tableau of this shape:

$$\begin{array}{cc} & \boxed{1} \\ \boxed{1} & \boxed{2} \end{array}.$$

Thus $\mathrm{SSYT}_n((2, 2)/(1)) \cong \mathrm{SSYT}_n((2, 1))$ when $n \geq 2$ and $s_{(2,2)/(1)} = s_{(2,1)}$.

Example 3.6. Next suppose $\lambda/\mu = (3, 2)/(1)$.

There are then two semistandard tableau of shape λ/μ with Yamanouchi reading words:

$$\begin{array}{ccc} & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{2} & \end{array} \quad\text{and}\quad \begin{array}{ccc} & \boxed{1} & \boxed{1} \\ \boxed{2} & \boxed{2} & \end{array}.$$

Thus $\mathrm{SSYT}_n((3, 2)/(1)) \cong \mathrm{SSYT}_n((3, 1)) \sqcup \mathrm{SSYT}_n((2, 2))$ when $n \geq 2$ and $s_{(3,2)/(1)} = s_{(3,1)} + s_{(2,2)}$.

Skew tableaux arise when we consider branchings of $\mathrm{SSYT}_n(\lambda)$ from type $\mathrm{GL}(n)$ to $\mathrm{GL}(r) \times \mathrm{GL}(n - r)$.

The weight lattice for type $\mathrm{GL}(n)$ is $\mathbb{Z}^n$ and the simple roots are $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i \in [n - 1]$.

The weight lattice for type $\mathrm{GL}(r) \times \mathrm{GL}(n-r)$ is also $\mathbb{Z}^n$ but the simple roots are now α_i for $i \in [n-1] \setminus \{r\}$.

We defined the direct product $\mathcal{B} \times \mathcal{C}$ of crystals for distinct Cartan types in Lecture 6.

Lemma 3.7. A connected Stembridge $\mathrm{GL}(r) \times \mathrm{GL}(n - r)$ crystal is isomorphic to a direct product $\mathcal{B} \times \mathcal{C}$ where $\mathcal{B}$ is a Stembridge $\mathrm{GL}(r)$ crystal and $\mathcal{C}$ is a Stembridge $\mathrm{GL}(n - r)$ crystal.

Proof. If our connected Stembridge $\mathrm{GL}(r)\times\mathrm{GL}(n-r)$ crystal has highest weight $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{Z}^n$ then it must be isomorphic to $\mathrm{SSYT}_r(\mu') \times \mathrm{SSYT}_{n-r}(\mu'')$ for $\mu' = (\mu_1, \dots, \mu_r)$ and $\mu'' = (\mu_{r+1}, \dots, \mu_n)$, as this product is another connected Stembridge $\mathrm{GL}(r) \times \mathrm{GL}(n - r)$ crystal with highest weight μ. □

Theorem 3.8. Let λ be a partition of m with $\ell(\lambda) \leq n$, that is, with at most n parts.

The crystal obtained by branching $\mathrm{SSYT}_n(\lambda)$ to type $\mathrm{GL}(r) \times \mathrm{GL}(n - r)$ is isomorphic to

$$\bigsqcup_{\substack{\mu \\ \mathsf{D}_\mu \subseteq \mathsf{D}_\lambda}} \mathrm{SSYT}_r(\mu) \times \mathrm{SSYT}_{n-r}(\lambda/\mu) \cong \bigsqcup_{\substack{\mu,\nu \\ |\mu|+|\nu|=m \\ \ell(\mu)\leq n \\ \ell(\nu)\leq n-r}} (\mathrm{SSYT}_r(\mu) \times \mathrm{SSYT}_{n-r}(\nu))^{\otimes c^\lambda_{\mu\nu}}.$$

Proof. Given $T \in \mathrm{SSYT}_n(\lambda)$, let μ be the partition whose Young diagram D_μ contains precisely the positions (i,j) with $T_{ij} \leq r$, so that $(i,j) \in \mathsf{D}_{\lambda/\mu}$ if and only if $T_{ij} > r$. Decomposing T as the union of a semistandard tableau U of shape μ and a semistandard skew tableau V of shape λ/μ (and then subtracting r from each entry of V) gives a bijection

$$\mathrm{SSYT}_n(\lambda) \to \bigsqcup_{\mathsf{D}_\mu \subseteq \mathsf{D}_\lambda} \mathrm{SSYT}_r(\mu) \times \mathrm{SSYT}_{n-r}(\lambda/\mu)$$

which one can check is actually a morphism of $\mathrm{GL}(r) \times \mathrm{GL}(n-r)$ crystals.

The right hand expansion is the result of applying Proposition 3.3. □