1 Review from last time

1.1 The plactic monoid

Elements $x_1 \in \mathcal{C}_1$ and $x_2 \in \mathcal{C}_2$ of normal crystals of the same Cartan type are *plactically equivalent* if the full subcrystals they belong to are isomorphic and the unique isomorphism between them maps $x_1 \mapsto x_2$.

In this case we write $x_1 \equiv x_2$. If $x_1 \equiv x_2$ and $y_1 \equiv y_2$ then $x_1 \otimes y_1 \equiv x_2 \otimes y_2$.

For a given normal crystal $\mathcal{B}$, the tensor product $\otimes$ gives the set $\mathsf{Plactic}(\mathcal{B})$ of plactic equivalence classes in the union $\{\emptyset\} \sqcup \mathcal{B} \sqcup (\mathcal{B} \otimes \mathcal{B}) \sqcup (\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}) \sqcup \cdots$ a monoid structure.

In type $\mathrm{GL}(n)$, plactic equivalence on words $w_1 w_2 \cdots w_m \in \mathbb{B}_n^{\otimes m}$ is the same thing as *Knuth equivalence.*

Moreover, $\mathsf{Plactic}(\mathbb{B}_n)$ is isomorphic to the monoid SSYT_n consisting of all semistandard tableaux with entries in $\{1, 2, \dots, n\}$, whose product is $U \circ V = P_{\mathsf{RSK}}(\mathfrak{row}(U)\mathfrak{row}(V))$.

1.2 Yamanouchi words

A word $w = w_1 w_2 \cdots w_m$ is a *Yamanouchi word* if for each $i > 0$ all of the final segments $w_{k+1}w_{k+2}\cdots w_m$ contain at least as many letters equal to i as $i+1$. For example : 153423211 but not 253423211.

Last time: a word $w \in \mathbb{B}_n^{\otimes m}$ is a highest weight element if and only if it is a *Yamanouchi word.*

1.3 Skew tableaux

A *skew shape* is a pair of partitions λ/μ with $\mathsf{D}_\lambda \subseteq \mathsf{D}_\mu := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq j \leq \mu_i\}$.

A *skew tableau* of shape λ/μ is a map $T : \mathsf{D}_{\lambda/\mu} \to \{1, 2, 3, \dots\}$ where $\mathsf{D}_{\lambda/\mu} := \mathsf{D}_\lambda \setminus \mathsf{D}_\mu$.

Such a map is *semistandard* if its rows are weakly increasing and its columns are strictly increasing. Let $\mathrm{SSYT}_n(\lambda/\mu)$ denote the set of all semistandard skew tableaux of shape λ/μ with entries in $\{1, 2, \dots, n\}$.

The *reading word* $\mathfrak{row}(T)$ of a skew tableau is defined in the same way as for an ordinary tableau. The set of words $\mathfrak{row}(T) \in \mathbb{B}_n^{\otimes|\lambda|-|\mu|}$ for $T \in \mathrm{SSYT}_n(\lambda/\mu)$ is a subcrystal, so there is a unique $\mathrm{GL}(n)$ crystal structure on $\mathrm{SSYT}_n(\lambda/\mu)$ such that $\mathfrak{row} : \mathrm{SSYT}_n(\lambda/\mu) \to \mathbb{B}_n^{\otimes m}$ is a crystal morphism.

The character of this crystal is the *skew Schur polynomial* $s_{\lambda/\mu}(x_1, x_2, \dots, x_n) := \sum_{T \in \mathrm{SSYT}_n(\lambda/\mu)} x^{\mathbf{wt}(T)}$.

Let $c_{\mu\nu}^\lambda$ be the number of semistandard skew tableau of shape λ/μ with weight ν whose reading words are Yamanouchi words. If $n \gg 0$ then we have $\mathrm{SSYT}_n(\lambda/\mu) \cong \bigsqcup_\nu \mathrm{SSYT}_n(\nu)^{\otimes c_{\mu\nu}^\lambda}$.

It follows that $s_{\lambda/\mu} = \sum_\nu c_{\mu\nu}^\lambda s_\nu$. One can also show that $s_\mu s_\lambda = \sum_\nu c_{\lambda\mu}^\nu s_\nu$.

2 Representations of $\mathrm{GL}(n, \mathbb{C})$

This lecture corresponds to Appendix A in Bump and Schilling's book. The goal is to connect some of the crystal combinatorics from last time to the representation theory of reductive complex Lie groups.

Let $G = \mathrm{GL}(n, \mathbb{C})$ be the general linear group of invertible $n \times n$ complex matrices.

If V is any finite-dimensional vector space then let $\mathrm{GL}(V)$ denote the group of linear bijections $V \to V$. Choosing a basis determines an isomorphism $\mathrm{GL}(V) \cong \mathrm{GL}(\dim V, \mathbb{C})$.

A *finite-dimensional representation* of G is a pair (π, V) where V is a finite-dimensional complex vector space and $\pi : G \to \mathrm{GL}(V)$ is a homomorphism that is *regular* in the sense that if $g = (g_{ij}) \in G$ is a matrix and we identify $\mathrm{GL}(V) = \mathrm{GL}(\dim V, \mathbb{C})$ by choosing any basis so that $\pi(g) = (\pi(g)_{kl})$ is another matrix, then the matrix coefficients $\pi(g)_{kl}$ can be written as polynomials in the g_{ij} and $\det(g)^{-1}$.

If the matrix coefficients $\pi(g)_{kl}$ do not involve $\det(g)^{-1}$ then (π, V) is a *polynomial representation.*

Example 2.1. The pair $(\det, \mathbb{C})$ is a finite-dimensional representation, as is $(\det^{-1}, \mathbb{C})$.

Only the first of these is a polynomial representation.

If (π, V) is a finite-dimensional representation then so is $(\det^N \otimes \pi, V)$ for any integer N.

If $N \gg 0$ is sufficiently large then $(\det^N \otimes \pi, V)$ will be a polynomial representation.

Remark. Let $\mathrm{Mat}_n(\mathbb{C})$ be the affine algebraic variety of all $n \times n$ complex matrices.

The group $\mathrm{GL}(n, \mathbb{C})$ is the open subvariety of $\mathrm{Mat}_n(\mathbb{C})$ given by $\{g \in \mathrm{Mat}_n(\mathbb{C}) : \det(g) \neq 0\}$.

This means that $\mathrm{GL}(n, \mathbb{C})$ is itself an affine variety, and it follows by general considerations that its coordinate ring is obtained from the coordinate ring of $\mathrm{Mat}_n(\mathbb{C})$ by adjoining $\det^{-1}$.

The coordinate ring of $\mathrm{Mat}_n(\mathbb{C})$ is the polynomial ring $\mathbb{Z}[x_{ij} : 1 \leq i, j \leq n]$ where $x_{ij} : g \mapsto g_{ij}$.

A map $\phi : X \to Y$ between affine algebraic varieties if *regular* if $f \circ \phi \in \mathcal{O}(X)$ for all $f \in \mathcal{O}(Y)$, where $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are the coordinate rings. Our requirements for representations $\pi : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(m, \mathbb{C})$ just mean that π is regular as a map between affine varieties $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{Mat}_m(\mathbb{C})$.

If (π, V) is a representation then V is a G-module for the action $gv := \pi(g)v$.

The representation (π, V) is *irreducible* if this G-module is irreducible, that is, if V is nonzero and has no proper nonzero subspaces that are G-invariant.

Two representations (π, V) and (π', V') are isomorphic if there is a linear bijection $\phi : V \to V'$ such that

$$\begin{array}{ccc} V & \xrightarrow{\pi(g)} & V \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ V' & \xrightarrow{\pi'(g)} & V' \end{array}$$

is a commutative diagram for all $g \in G = \mathrm{GL}(n, \mathbb{C})$.

Given a finite-dimensional complex vector space V, let V^* be the vector space of linear maps $\lambda : V \to \mathbb{C}$.

If (π, V) is a finite-dimensional representation then let $\hat{\pi} \in \mathrm{GL}(V^*)$ be the map

$$\hat{\pi}(g) : \lambda \mapsto \lambda \circ \pi(g^{-1}).$$

The pair $(\hat{\pi}, V^*)$ is then another finite-dimensional representation, called the *dual* or *contragredient representation.* This may not be isomorphic to (π, V); when it is, we say that (π, V) is *self-dual.*

3 Lie algebras

The Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ of $\mathrm{GL}(n, \mathbb{C})$ is the set $\mathrm{Mat}_n(\mathbb{C})$ with the Lie bracket $[X, Y] = XY - YX$.

If (π, V) is a finite-dimensional representation then $\mathfrak{gl}(n, \mathbb{C}) = \mathrm{Mat}_n(\mathbb{C})$ acts on V by the formula

$$Xv := \tfrac{d}{dh}\pi(e^{hX})v\Big|_{h=0} \qquad \text{where } e^X := \sum_{k=0}^{\infty} X^k/k! \in \mathrm{GL}(n, \mathbb{C}) \text{ for a square matrix } X.$$

A matrix $g \in \mathrm{GL}(n, \mathbb{C})$ is *unitary* if its inverse $g^{-1} = \overline{g}^T$ is given by its conjugate transpose.

The *unitary group* $U(n)$ consists of all unitary matrices in $\mathrm{GL}(n,\mathbb{C})$.

This group is a closed and bounded (and therefore compact) subset of $\mathrm{Mat}_n(\mathbb{C})$.

In fact, $U(n)$ is the maximal compact subgroup of $\mathrm{GL}(n,\mathbb{C})$.

If $X \in \mathfrak{gl}(n,\mathbb{C}) = \mathrm{Mat}_n(\mathbb{C})$ then $e^X \in U(n)$ if and only if $e^{-X} = e^{\overline{X}^T}$, i.e., if $X = -\overline{X}^T$ is *skew-hermitian*.

Let $\mathfrak{u}(n) \subset \mathfrak{gl}(n,\mathbb{C})$ be the Lie subalgebra of skew-hermitian matrices. This is the Lie algebra of $U(n)$.

4 Weight spaces and roots

Let $T = T(n,\mathbb{C})$ be the subgroup of diagonal matrices in $G = \mathrm{GL}(n,\mathbb{C})$.

A *character* of T is a homomorphism $T \to \mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times$. A polynomial character of T must be a map of the form

$$t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \prod_{i=1}^n t_i^{a_i} \qquad \text{for some nonnegative integers } a_i \in \mathbb{N}.$$

Regular characters of T can also involve $\det^{-N}$ and so are maps of the form $t \mapsto (t_1 t_2 \cdots t_n)^{-N} \prod_{i=1}^n t_i^{a_i}$ where $a_i \in \mathbb{N}$ and $N \in \mathbb{Z}$. Thus each regular character of T can be expressed as

$$t \mapsto t^\mu := \prod_{i=1}^n t_i^{\mu_i} \quad \text{for a unique vector } \mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n.$$

We refer to regular characters of T as *weights*. We identify the set of weights with $\mathbb{Z}^n$.

Note: when discussing characters of crystals, "t^μ" was just a formal symbol; now this stands for a specific complex number, given by the value of a weight applied to $t \in T$.

Let (π, V) be a finite-dimensional representation of $\mathrm{GL}(n,\mathbb{C})$. The *weight space* of $\mu \in \mathbb{Z}^n$ is

$$V_\mu := \{v \in V : \pi(t)v = t^\mu v \text{ for all } t \in T\}.$$

The vector μ is a *weight* of the representation π if $V_\mu \neq 0$. We have $V = \bigoplus_{\mu \in \mathbb{Z}^n} V_\mu$.

The *adjoint representation* of $G = \mathrm{GL}(n,\mathbb{C})$ is $(\mathrm{Ad}, \mathrm{Mat}_n(\mathbb{C}))$ where $\mathrm{Ad}(g)X = gXg^{-1}$.

The *roots* are the weights of the adjoint representation. The set of roots forms a root system Φ.

If $X = E_{ij}$ is the elementary matrix with 1 in position (i,j) then $\mathrm{Ad}(t)X = t^{\mathbf{e}_i - \mathbf{e}_j} X$ for all $i \neq j$.

It follows that $\Phi = \{\mathbf{e}_i - \mathbf{e}_j : 1 \leq i, j \leq n\}$ as usual, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbb{Z}^n$ is the standard basis.

Define $X_{\mathbf{e}_i - \mathbf{e}_j} = E_{ij} \in \mathrm{Mat}_n(\mathbb{C})$ and define $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1} \in \Phi$.

Again (π, V) be a finite-dimensional representation of $\mathrm{GL}(n,\mathbb{C})$.

Recall that $X \in \mathrm{Mat}_n(\mathbb{C})$ acts on V by $Xv = \frac{d}{dh}\pi(e^{hX})v\Big|_{h=0}$.

Proposition 4.1. If $\alpha \in \Phi$ then the action of X_α maps $V_\mu \mapsto V_{\mu+\alpha}$.

Proof. Write gv instead of $\pi(g)v$ for $g \in G$ and $v \in V$. Also let $X = X_\alpha$.

Then for $t \in T$ and $v \in V_\mu$ we have

$$tXv = \frac{d}{dh} e^{htX} v\Big|_{h=0} = \frac{d}{dh} e^{htXt^{-1}} tv\Big|_{h=0} = \frac{d}{dh} e^{ht^\alpha X} t^\mu v\Big|_{h=0} = t^{\mu+\alpha} Xv$$

where the last step follows by the chain rule, which is justified since all functions here are analytic. ☐

The *root operators* $E_i := X_{\alpha_i}$ and $F_i := X_{-\alpha_i}$ are the analogues of the raising and lowering operators e_i and f_i for $\mathrm{GL}(n)$ crystals.

5 Complete reducibility

The understand finite-dimensional representations of $\mathrm{GL}(n, \mathbb{C})$ we can just consider irreducible ones.

Proposition 5.1 (Weyl's unitarian trick)**.** Let (π, V) be a finite-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$. Any $U(n)$-invariant subspace $W \subseteq V$ is also $\mathrm{GL}(n, \mathbb{C})$-invariant. Therefore V is irreducible as a $\mathrm{GL}(n, \mathbb{C})$-module if and only if V is irreducible as a $U(n)$-module.

Proof. If $W \subseteq V$ is invariant under $U(n)$ then it is invariant under the action of $\mathfrak{u}(n)$.

But this action $(X, v) \mapsto Xv = \frac{d}{dh}\pi(e^{hX})v\Big|_{h=0}$ is linear in X as well as v, so since $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$ it follows that W is invariant under the action of $\mathfrak{gl}(n, \mathbb{C})$.

Exponentiating shows that W is therefore $\mathrm{GL}(n, \mathbb{C})$-invariant. □

If (π, V) and (π', V') are representations, then so is the direct sum $(\pi, V) \oplus (\pi', V') := (\pi \oplus \pi', V \oplus V')$.

Proposition 5.2. Each finite-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$ is isomorphic to a direct sum of irreducible representations.

Proof. By the unitarian trick, it suffices to show that each finite-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$ is isomorphic to a direct sum of representations that are irreducible for $U(n)$.

Since $U(n)$ is compact, we can always find a $U(n)$-invariant inner product on a representation: take any nondegenerate bilinear form then average over the group by integrating.

Any nonzero invariant subspace of minimal dimension in our representation is irreducible, and its orthogonal complement is then $U(n)$-invariant and decomposes into a direct irreducible subrepresentations by induction on dimension. □

6 Characters

Suppose (π, V) is a finite-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$.

We say that π is *homogeneous of degree* k if (π, V) is a polynomial representation and the coefficients $\pi(g)_{kl}$ are homogeneous polynomials of degree k in the matrix entries g_{ij}.

Any irreducible polynomial representation is homogeneous, so we can decompose any polynomial representation as a direct sum of homogeneous representations.

The *character* $\chi_\pi : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}$ of π is $\chi_\pi(g) = \mathrm{tr}(\pi(g))$, which is a polynomial in g_{ij} and $\det^{-1}(g)$.

The *weight multiplicity* of $\mu \in \mathbb{Z}^n$ in V is $\dim(V_\mu)$, and we have

$$\chi_\pi(t) = \sum_{\mu \in \mathbb{Z}^n} \dim(V_\mu) t^\mu \qquad \text{for } t \in T.$$

Let $N(T) = \{g \in \mathrm{GL}(n, \mathbb{C}) : gTg^{-1} = T\}$ be the normalizer of T.

This is the subgroup of *monomial matrices*, i.e., matrices with exactly one nonzero entry in each row and column. We identify the quotient $W := N(T)/T$ with the symmetric group S_n. This group acts on $\mathbb{Z}^n$ by permuting coordinates. One can check that $\dim(V_\mu)$ is constant on the orbits of $\mathbb{Z}^n$ under this action.

Recall that a weight $\lambda \in \mathbb{Z}^n$ is *dominant* if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

A dominant weight is a *partition* is $\lambda_n \geq 0$.

The *dominance order* $\preceq$ is the partial order on $\mathbb{Z}^n$ that has $\mu \preceq \lambda$ if

$$\mu_1 \leq \lambda_1, \qquad \mu_1 + \mu_2 \leq \lambda_1 + \lambda_2, \qquad \mu_1 + \mu_2 + \mu_3 \leq \lambda_1 + \lambda_2 + \lambda_3, \quad \dots$$

If Λ^+ is the set of dominant weights then $\mu \preceq \lambda$ if and only if $\langle \lambda - \mu, \nu \rangle \geq 0$ for all $\nu \in \Lambda^+$.

If (π, V) is a finite-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$ then a *maximal weight* for π is a weight $\lambda \in \mathbb{Z}^n$ that is maximal under the dominance order. This means that $\lambda \in \mathbb{Z}^n$ is a maximal weight for π if and only if $V_\lambda \neq 0$ but $V_\mu = 0$ whenever $\lambda \prec \mu$.

If (π, V) is irreducible then a maximal weight is called a *highest weight*. If (π, V) is reducible then a *highest weight* is a maximal weight that is the highest weight of an irreducible subrepresentation.

Every finite-dimensional representation has at least one maximal weight.

Lemma 6.1. A maximal weight for a finite-dimensional representation (π, V) is dominant.

Proof. Given a maximal weight $\lambda \in \mathbb{Z}^n$, if λ is not dominant then $\lambda_i < \lambda_{i+1}$ for some index i. Let $\mu = s_i(\lambda)$. Then $\dim(V_\mu) = \dim(V_\lambda) \neq 0$ but $\lambda \prec \mu$, contradicting maximality. □

Theorem 6.2 (Weyl character formula)**.** Let (π, V) be an irreducible finite-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$. Then π has a unique highest weight $\lambda \in \mathbb{Z}^n$, and $\dim(V_\lambda) = 1$. Moreover:

(a) Any other irreducible finite-dimensional representation with highest weight λ is isomorphic to (π, V).

(b) Every dominant weight is the highest weight of some irreducible representation.

(c) If $t \in T$ then the value of the character $\chi_\pi(t)$ has the formula

$$\chi_\pi(t) = \frac{\sum_{w \in W} \mathrm{sgn}(w) t^{w(\lambda+\rho)}}{\sum_{w \in W} \mathrm{sgn}(w) t^{w(\rho)}}$$

where $W = S_n$, $\rho = (n-1, n-2, \dots, 0) \in \mathbb{Z}^n$, and $\mathrm{sgn}(w) \in \{\pm 1\}$ is the sign a permutation.

This result is worth remembering but we won't give a proof, which is out of scope for this lecture.

Let $\pi_\lambda^{\mathrm{GL}(n)}$ denote an irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ with highest weight λ.

This is a (homogeneous) polynomial representation if and only if λ is a partition.

Assume we are in this case. Then the character of $\pi = \pi_\lambda^{\mathrm{GL}(n)}$ evaluated at $t \in T$ can be written as a homogeneous polynomial in the diagonal entries $t_1, t_2, \dots, t_n$. The Weyl character formula for $\mathrm{GL}(n, \mathbb{C})$ says that this polynomial is precisely the *Schur polynomial* $s_\lambda(x_1, x_2, \dots, x_n)$. In detail, one can rewrite the Weyl character formula as a quotient of determinants, and this gives the right hand side of

$$s_\lambda(t_1, t_2, \dots, t_n) = \frac{\det \left[t_i^{\lambda_j + n - 1 - j} \right]_{1 \leq i,j \leq n}}{\det \left[t_i^{n-1-j} \right]_{1 \leq i,j \leq n}}$$

which is another well-known definition of the Schur polynomials.

Observe that the Vandermonde determinant formula is $\det \left[t_i^{n-1-j} \right]_{1 \leq i,j \leq n} = \prod_{1 \leq i < j \leq n} (t_i - t_j)$.

The previous theorem can be viewed as the (type A) representation analogue of our main theorem about normal crystals. We conclude today with two other analogies with crystals:

Proposition 6.3. Let λ and μ be dominant elements of $\mathbb{Z}^n$.

Then $\pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\mu^{\mathrm{GL}(n)}$ has a unique irreducible subrepresentation isomorphic to $\pi_{\lambda+\mu}^{\mathrm{GL}(n)}$.

Proof. If $\dim_\lambda(\nu)$ denotes the dimension of the ν weight space for the representation $\pi_\lambda^{\mathrm{GL}(n)}$, then the dimension of the ν weight space for $\pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\mu^{\mathrm{GL}(n)}$ must have dimension $\sum_{\nu=\nu_1+\nu_2} d_\lambda(\nu_1)d_\mu(\nu_2)$.

Since λ and μ are the unique maximal weights for $\pi_\lambda^{\mathrm{GL}(n)}$ and $\pi_\mu^{\mathrm{GL}(n)}$, it follows that $\lambda+\mu$ is the unique maximal weight for $\pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\mu^{\mathrm{GL}(n)}$, and that the corresponding weight space has dimension one.

From this observation, the result follows by the Weyl character formula theorem. $\square$

Proposition 6.4. Let λ be any dominant weight for $\mathrm{GL}(n,\mathbb{C})$ that is a partition of k.

Then $\pi_\lambda^{\mathrm{GL}(n)}$ is isomorphic to a subrepresentation of the $\mathrm{GL}(n,\mathbb{C})$-module $(\mathbb{C}^n)^{\otimes k}$.

Proof. One can show that if $\lambda = (1^k)$ then $\pi_\lambda^{\mathrm{GL}(n)}$ is isomorphic to the k-th exterior power $\bigwedge^k(\mathbb{C}^n)$ which is a summand of $(\mathbb{C}^n)^{\otimes k}$. Any other dominant weight that is a partition can be expressed as a nonnegative integer linear combination of the fundamental weights $\varpi_k = (1^k) = (1,1,\dots,1,0,0,\dots,0) \in \mathbb{Z}^n$. Using this fact, the result follows by repeatedly applying the previous proposition. $\square$

We can view $\mathrm{SSYT}_n(\lambda)$ as the crystal analogue of the representation $\pi_\lambda^{\mathrm{GL}(n)}$, and this proposition corresponds to how $\mathrm{SSYT}_n(\lambda)$ is a full subcrystal of $\mathbb{B}_n^{\otimes|\lambda|}$.