1 Review from last time: representations of $\mathrm{GL}(n,\mathbb{C})$

Everywhere in this lecture, n is a fixed positive integer and $\mathbb{C}$ is the field of complex numbers.

A *finite-dimensional representation* of $\mathrm{GL}(n,\mathbb{C})$ is a pair (π, V) where V is a finite-dimensional complex vector space and $\pi : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(V)$ is a homomorphism that is *regular* as a map of affine varieties. Concretely, this means that if $g = (g_{ij}) \in \mathrm{GL}(n,\mathbb{C})$ is a matrix and we identify $\mathrm{GL}(V) = \mathrm{GL}(\dim V, \mathbb{C})$ so that $\pi(g) = (\pi(g)_{kl})$ is another matrix, then the coefficients $\pi(g)_{kl}$ are polynomials in g_{ij} and $\det(g)^{-1}$.

If the matrix coefficients $\pi(g)_{kl}$ do not involve $\det(g)^{-1}$ then (π, V) is a *polynomial representation*.

Each finite-dimensional representation is isomorphic to a direct sum of irreducible representations.

Let (π, V) be a finite-dimensional representation of $\mathrm{GL}(n,\mathbb{C})$.

Let T be the subgroup of matrices $t = \mathrm{diag}(t_1, t_2, \dots, t_n) \in \mathrm{GL}(n,\mathbb{C})$ and set $t^\mu = \prod_{i=1}^n t_i^{\mu_i}$ for $\mu \in \mathbb{Z}^n$.

The *weight space* of $\mu \in \mathbb{Z}^n$ in (π, V) is $V_\mu := \{v \in V : \pi(t)v = t^\mu v \text{ for all } t \in T\}$.

The vector $\mu \in \mathbb{Z}^n$ is a *weight* of the representation π if $V_\mu \neq 0$. We have $V = \bigoplus_{\mu \in \mathbb{Z}^n} V_\mu$.

A weight $\lambda \in \mathbb{Z}^n$ for π is *maximal* if it is dominant and maximal under the dominance order.

Dominant means $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The *character* $\chi_\pi : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}$ of π is $\chi_\pi(g) = \mathrm{tr}(\pi(g))$.

Weyl character formula: Assume (π, V) is an irreducible finite-dimensional representation of $\mathrm{GL}(n,\mathbb{C})$.

- Then (π, V) has a unique maximal weight $\lambda \in \mathbb{Z}^n$ (called the *highest weight*), and $\dim(V_\lambda) = 1$.
- Any other irreducible finite-dimensional representation with highest weight λ is isomorphic to (π, V).
- Every dominant weight is the highest weight of some irreducible representation.
- If $t \in T$ then $\chi_\pi(t) = \frac{\sum_{w \in S_n} \mathrm{sgn}(w) t^{w(\lambda+\rho)}}{\sum_{w \in S_n} \mathrm{sgn}(w) t^{w(\rho)}}$ where $\rho = (n-1, \dots, 2, 1, 0)$.

  Caution: if t has repeated eigenvalues (e.g., if $t = 1$) then this formula becomes $\chi_\pi(t) = \frac{0}{0}$, but we can extract the correct character value by interpreting the ratio as a limit.

For each dominant $\lambda \in \mathbb{Z}^n$, write $\pi_\lambda^{\mathrm{GL}(n)}$ for the irreducible representation with highest weight λ.

Weyl character formula $\Rightarrow$ if $\lambda_n \geq 0$ then $\chi_\pi(t) = s_\lambda(t_1, \dots, t_n)$ for $\pi = \pi_\lambda^{\mathrm{GL}(n)}$ and $t = \mathrm{diag}(t_1, \dots, t_n)$.

Analogous to crystals, if $\lambda \in \mathbb{Z}^n$ is a partition then $\pi_\lambda^{\mathrm{GL}(n)}$ is isomorphic to a subrepresentation of $(\mathbb{C}^n)^{\otimes|\lambda|}$.

2 Commuting endomorphism rings

The irreducible polynomial representations of $\mathrm{GL}(n,\mathbb{C})$ are indexed by partitions λ with $\ell(\lambda) \leq n$.

On the other hand, the irreducible representations of S_k are also indexed by partitions λ with $|\lambda| = k$.

This is not a coincidence, and there is a canonical correspondence between the two families of representations that explains how both should by labeled by partitions in a consistent way. This correspondence is called *Schur-Weyl duality* or sometimes *Frobenius-Schur duality* or *Schur duality*.

Fix positive integers k and n. Write $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ for the standard basis of $\mathbb{C}^n$.

The *standard module* of $\mathrm{GL}(n,\mathbb{C})$ is $\mathbb{C}^n$ with the action $g : v \mapsto gv$ multiplying a matrix and a vector.

The tensor power $(\mathbb{C}^n)^{\otimes k}$ is the n^k-dimensional vector space with basis elements $\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \dots \otimes \mathbf{e}_{i_k}$.

The elements of $(\mathbb{C}^n)^{\otimes k}$ are linear combinations of tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ with $v_i \in \mathbb{C}^n$, where

$$v_1 \otimes \cdots \otimes (av_i + bw_i) \otimes \cdots \otimes v_k = a(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k) + b(v_1 \otimes \cdots \otimes w_i \otimes \cdots \otimes v_k)$$
$$v_1 \otimes \cdots \otimes av_i \otimes v_{i+1} \otimes \cdots \otimes v_k = v_1 \otimes \cdots \otimes v_i \otimes av_{i+1} \otimes \cdots \otimes v_k$$

for any scalars $a, b \in \mathbb{C}$ and indices i. These relations generate the kernel of the map $(\mathbb{C}^n)^{\times k} \to (\mathbb{C}^n)^{\otimes k}$.

The group $\mathrm{GL}(n, \mathbb{C})$ acts linearly on $(\mathbb{C}^n)^{\otimes k}$ on the left by the formula

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k \qquad \text{for } g \in \mathrm{GL}(n, \mathbb{C}).$$

The symmetric group S_k acts linearly on $(\mathbb{C}^n)^{\otimes k}$ on the right by the formula

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_k)w = v_{w(1)} \otimes v_{w(2)} \otimes \cdots \otimes v_{w(k)} \qquad \text{for } w \in S_k.$$

These action makes $(\mathbb{C}^n)^{\otimes k}$ into a left $\mathrm{GL}(n, \mathbb{C})$-module and a right S_k-module. Since we have

$$(g(v_1 \otimes v_2 \otimes \cdots \otimes v_k))w = g((v_1 \otimes v_2 \otimes \cdots \otimes v_k)w) = gv_{w(1)} \otimes gv_{w(2)} \otimes \cdots \otimes gv_{w(k)},$$

these module structures are compatible and so $(\mathbb{C}^n)^{\otimes k}$ is a $\mathrm{GL}(n, \mathbb{C}) \times S_k$-module.

Suppose Ω is a vector space.

Let $\mathrm{End}(\Omega)$ denote the ring of all linear maps $\Omega \to \Omega$ and suppose $A \subseteq \mathrm{End}(\Omega)$ is a subring.

The *commuting ring* of A is the ring of linear maps $\Omega \to \Omega$ that commute with all elements of A.

Theorem 2.1. Let A and B be the subrings of $\mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right)$ generated as

$$A = \langle \lambda_g : g \in \mathrm{GL}(n, \mathbb{C}) \rangle \qquad \text{and} \qquad B = \langle \rho_w : w \in S_k \rangle$$

where λ_g and ρ_w are the linear maps $(\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes k}$ defined by

$$\lambda_g(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k \qquad \text{and} \qquad \rho_w(v_1 \otimes \cdots \otimes v_k) = v_{w(1)} \otimes \cdots \otimes v_{w(k)}.$$

Then B is the commuting ring of A and A is the commuting ring of B.

Proof. Let $V = \mathbb{C}^n$ and $\Omega = V^{\otimes k}$. We first show that A is the commuting ring of B.

We have already observed that the elements of A and B commute with each other. Define a linear map

$$\beta : \mathrm{End}(V) \times \cdots \times \mathrm{End}(V) \to \mathrm{End}(\Omega)$$

by letting $\beta(f_1, \dots, f_k)$ be the linear map sending $v_1 \otimes \cdots \otimes v_k \mapsto f_1(v_1) \otimes \cdots \otimes f_k(v_k)$.

A basis for $\mathrm{End}(V)$ is given by the linear maps indexed by pairs (i, j) that send $\mathbf{e}_i \mapsto \mathbf{e}_j$ and $\mathbf{e}_k \mapsto 0$ for $k \neq i$. By taking $f_1, \dots, f_k$ to be maps of this form one sees that $\mathrm{End}(\Omega)$ is spanned by the image of β.

Next, we observe that $\rho_w \circ \beta(f_1, \dots, f_k) \circ \rho_{w^{-1}} = \beta(f_{w(1)}, \dots, f_{w(k)})$.

It follows that the commuting ring of B is the span of the elements

$$\tilde{\beta}(f_1, \dots, f_k) := \frac{1}{k!} \sum_{w \in S_k} \beta(f_{w(1)}, \dots, f_{w(k)})$$

since if $\phi \in \mathrm{End}(\Omega)$ commutes with every ρ_w then $\phi = \frac{1}{k!} \sum_{w \in S_k} \rho_w \circ \phi \circ \rho_{w^{-1}}$.

The next step to check is that if $f_1, f_2, \dots, f_k \in \mathrm{End}(V)$ are fixed and $f_I := \sum_{i \in I} f_i$ for $I \subseteq [k]$, then

$$\tilde{\beta}(f_1, f_2, \dots, f_k) = \frac{1}{k!} \sum_{I \subseteq [k]} (-1)^{k-|I|} \beta(f_I, f_I, \dots, f_I).$$

This is a straightforward exercise using the inclusion-exclusion principle; we will skip the details.

We conclude that the commuting ring of B is spanned by the maps $\beta(f, f, \dots, f)$ where $f \in \mathrm{End}(V)$.

This is a promising observation because if $f \in \mathrm{GL}(V) \subset \mathrm{End}(V)$ then $\beta(f, f, \dots, f) = \lambda_f \in A$.

Suppose $f \in \mathrm{End}(V)$ is not invertible. We claim that we still have $\beta(f, f, \dots, f) \in A$ since this endomorphism is generated by elements of the form $\lambda_g \in A$ for $g \in \mathrm{GL}(V)$. One argument goes as follows.

Since $\det(f) = 0$ and the determinant is a nonzero polynomial function, if we choose $\epsilon > 0$ to be any sufficiently small real number, then $\det(f - \epsilon I) \neq 0$ and we have

$$g := f - \epsilon I \in \mathrm{GL}(V) \quad \text{and} \quad h := \epsilon I \in \mathrm{GL}(V) \quad \text{and} \quad f = g + h.$$

Let $\zeta \in \mathbb{C}$ be a primitive kth root of unity and define $Q(\phi) = \beta(\phi, \phi, \dots, \phi)$ for $\phi \in \mathrm{End}(V)$.

Note again that $Q(\phi) = \lambda_\phi \in A$ if ϕ is invertible.

If ϵ is small enough then we can further assume that $g + \zeta^e h \in \mathrm{GL}(V)$ for all $e \in \{1, 2, \dots, k-1\}$.

It is now enough to show that $\sum_{e=0}^{k-1} Q(g + \zeta^e h) = kQ(g) + kQ(h)$, since this implies that

$$Q(f) = Q(g+h) = kQ(g) + kQ(h) - \sum_{e=1}^{k-1} Q(g + \zeta^e h) = k\lambda_g + k\lambda_h - \sum_{e=1}^{k-1} \lambda_{g+\zeta^e h} \in A.$$

The $k = 2$ case of this identity is instructive. Then we have $\zeta_2 = -1$ and

$$\begin{aligned} Q(g+h) + Q(g-h) &= (g+h) \otimes (g+h) + (g-h) \otimes (g-h) \\ &= (g \otimes g + g \otimes h + h \otimes g + h \otimes h) + (g \otimes g - g \otimes h - h \otimes g + h \otimes h) \\ &= 2(g \otimes g + h \otimes h). \end{aligned}$$

Here we are writing "$\phi \otimes \psi$" to mean the map in $\mathrm{End}((\mathbb{C}^n)^2)$ with $v_1 \otimes v_2 \mapsto \phi(v_1) \otimes \psi(v_2)$.

For $k > 2$ a similar cancelation occurs because $1 + \zeta + \zeta^2 + \cdots + \zeta^{k-1} = 0$.

Specifically, consider a term in the expansion of $Q(g + \zeta^e h) = (g + \zeta^e h) \otimes (g + \zeta^e h) \otimes \cdots \otimes (g + \zeta^e h)$.

If this term corresponds to choosing g instead of $\zeta^e h$ in exactly m factors, then the coefficient is $\zeta^{e(k-m)}$.

We have $\sum_{e=0}^{k-1} \zeta^{e(k-m)} = 0$ if $m \notin \{0, k\}$. If $m \in \{0, k\}$ then this sum is k and the term is $Q(g)$ or $Q(h)$.

We conclude that $\sum_{e=0}^{k-1} Q(g + \zeta^e h) = kQ(g) + kQ(h)$ as needed.

Thus $A = \langle \lambda_g : g \in \mathrm{GL}(n, \mathbb{C}) \rangle$ is the commuting ring of $B = \langle \rho_w : w \in S_k \rangle$.

It remains to show the other statement in theorem, that B is likewise the commuting ring of A. The standard approach to this is not particularly constructive or self-contained.

The argument in Bump and Schilling's Appendix A.2 is to appeal to the fact that the ring B is *semisimple*, which lets us use the *Jacobson Density Theorem* to deduce that B is automatically the commuting ring of A since A is the commuting ring of B. A lot of prerequisites go into unpacking these claims.

For good discussions of more constructive proofs, check out these mathoverflow posts:

- Direct proof that the centralizer of $\mathrm{GL}(V)$ acting on $V^{\otimes n}$ is spanned by S_n
- How to constructively/combinatorially prove Schur-Weyl duality?

□

3 Schur-Weyl duality

To summarize: we have a left diagonal action of $\mathrm{GL}(n,\mathbb{C})$ on $(\mathbb{C}^n)^{\otimes k}$ which generates a ring of endomorphisms, and a right permutation action of S_k on $(\mathbb{C}^n)^{\otimes k}$ which generates another ring of endomorphisms, and each of these rings consists of precisely the endomorphisms of $(\mathbb{C}^n)^{\otimes k}$ that commute with all elements of the other ring. This fact leads to a consistent labeling of the irreducible (polynomial) representations of $\mathrm{GL}(n,\mathbb{C})$ and S_k via the following general proposition:

Proposition 3.1. Let Ω be a finite-dimensional vector space and let A and B be subalgebras of $\mathrm{End}(\Omega)$. Assume that A is the commuting ring of B and B is the commuting ring of A, so that the action

$$(\alpha,\beta)\cdot\omega := \alpha(\beta(\omega)) = \beta(\alpha(\omega)) \qquad \text{for } (\alpha,\beta)\in A\times B \text{ and } \omega\in\Omega$$

makes Ω into an $A\times B$-module. Suppose this module decomposes as $\Omega\cong\bigoplus_i U_i\otimes W_i$ where the U_i are A-modules and the W_i and B-modules. If $i\neq j$, then $U_i\not\cong U_j$ as A-modules and $W_i\not\cong W_j$ as B-modules.

Proof. Arguing by contradiction, suppose $U_i\cong U_j$ as A-modules for $i\neq j$ and let $\phi: U_i\to U_j$ be an A-module isomorphism. Let $\psi: W_i\to W_j$ be any nonzero linear map and define $f:\Omega\to\Omega$ to be the linear map that restricts to $\phi\otimes\psi$ on $U_i\otimes W_i$ and is zero on the other summands.

Then f commutes with all elements of A, so $f\in B$. But this is impossible since f does not map the B-module $U_i\otimes W_i$ to itself. Thus no such isomorphism $U_i\cong U_j$ can exist.

The argument that $W_i\not\cong W_j$ for $i\neq j$ is similar. □

Theorem 3.2. The $\mathrm{GL}(n,\mathbb{C})\times S_k$-module $(\mathbb{C}^n)^{\otimes k}$ decomposes as

$$(\mathbb{C}^n)^{\otimes k} = \bigoplus_\lambda V_\lambda^{\mathrm{GL}(n)}\otimes V_\lambda^{S_k}$$

where λ runs through all partitions of k with at most n parts, and:

- $V_\lambda^{\mathrm{GL}(n)}$ is an irreducible $\mathrm{GL}(n,\mathbb{C})$-module isomorphic to the representation $\pi_\lambda^{\mathrm{GL}(n)}$.
- $V_\lambda^{S_k}$ is a certain irreducible S_k-module that depends only on λ up to isomorphism.

Moreover, if $n\geq k$ then the modules $V_\lambda^{S_k}$ are a complete set of non-isomorphic irreducible S_k-modules.

Remark. It therefore makes sense to label the S_k-representation corresponding to $V_\lambda^{S_k}$ by $\pi_\lambda^{S_k}$.

One can check that this makes $\pi_{(k)}^{S_k}$ and $\pi_{(1^k)}^{S_k}$ the trivial and sign representations of S_k using the fact that $V_{(k)}^{\mathrm{GL}(n)}$ is the kth symmetric power of $\mathbb{C}^n$ while $V_{(1^k)}^{\mathrm{GL}(n)}$ is the kth exterior power of $\mathbb{C}^n$.

Proof. Each diagonal matrix $t=\mathrm{diag}(t_1,t_2,\dots,t_k)$ multiplies the vector $\mathbf{e}_{i_1}\otimes\mathbf{e}_{i_2}\otimes\cdots\otimes\mathbf{e}_{i_k}$ by $t_{i_1}t_{i_2}\cdots t_{i_k}$, so the weights of $(\mathbb{C}^n)^{\otimes k}$ must all be homogenous monomials of degree k.

This means that the irreducible $\mathrm{GL}(n,\mathbb{C})$-representations $V_\lambda^{\mathrm{GL}(n)}$ that appear in our decomposition must be indexed by some subset of partitions of k with at most n parts.

It follows from the preceding proposition with $V=\mathbb{C}^n$ and $\Omega=V^{\otimes k}$ that there are no repetitions among (the isomorphism classes of) the modules $V_\lambda^{\mathrm{GL}(n)}$ or $V_\lambda^{S_k}$. The fact that $V_\lambda^{\mathrm{GL}(n)}$ appears for every partition λ of k with at most n parts holds since we know that $\pi_\lambda^{\mathrm{GL}(n)}$ is a constituent of $(\mathbb{C}^n)^{\otimes k}$.

If $n\geq k$, then the number of such partitions is the same as the total number of partitions of k, which is the number of isomorphism classes of irreducible S_k-representations. One concludes that the non-isomorphic S_k-modules $V_\lambda^{S_k}$ must therefore represent all isomorphism classes of irreducible S_k-modules. □

Suppose G_1 and G_2 are groups and Ω is representation for the direct product $G_1 \times G_2$.

We say that Ω is a *correspondence* if there is a decomposition into irreducible $G_1 \times G_2$-subrepresentations

$$\Omega = \bigoplus_i \pi_i^{G_1} \otimes \pi_i^{G_2}$$

such that $\pi_i^{G_1} \not\cong \pi_j^{G_1}$ as G_1-representations if $i \neq j$ and $\pi_i^{G_2} \not\cong \pi_j^{G_2}$ as G_2-representations if $i \neq j$.

In this case, Ω determines a bijection between the set of G_1-representations $\{\pi_i^{G_1}\}$ and the set of G_2-representations $\{\pi_i^{G_2}\}$. We indicate this situation by writing

$$\pi_i^{G_1} \xleftrightarrow{\Omega} \pi_i^{G_2}.$$

For example, *Schur-Weyl duality* refers to the correspondence written as $\pi_\lambda^{\mathrm{GL}(n)} \xleftrightarrow{(\mathbb{C}^n)^{\otimes k}} \pi_\lambda^{S_k}$.

4 Symmetric functions

We have briefly encountered the *Schur functions* $s_\lambda := \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathbf{wt}(T)}$ which are the formal power series analogues of the Schur polynomials $s_\lambda(x_1, x_2, \dots, x_n) \in \mathsf{Sym}_n \subset \mathbb{Z}[x_1, x_2, \dots, x_n]$.

Let Sym be the abelian group of formal power series spanned by the linearly independent set of Schur functions s_λ as λ varies over all partitions. The results summarized at the start of Lecture 4 show that this abelian group is actually a graded ring. (For the grading, each s_λ is homogeneous of degree $|\lambda|$.)

The ring Sym can also be identified as the inverse limit of the system $\mathsf{Sym}_0 \twoheadleftarrow \mathsf{Sym}_1 \twoheadleftarrow \mathsf{Sym}_2 \twoheadleftarrow \cdots$ where the projection $\mathsf{Sym}_n \twoheadleftarrow \mathsf{Sym}_{n+1}$ is the map setting $x_{n+1} = 0$.

Let $e_k := s_{(1^k)} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ and $h_k := s_{(k)} = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ for $k \geq 1$.

These are the *elementary symmetric functions* and the *complete homogeneous symmetric functions.*

Theorem 4.1. It holds that $\mathsf{Sym} = \mathbb{Z}[e_1, e_2, e_3, \dots]$ and also $\mathsf{Sym} = \mathbb{Z}[h_1, h_2, h_3, \dots]$.

Consequently, there is a unique ring homomorphism $\omega : \mathsf{Sym} \to \mathsf{Sym}$ with $\omega(e_k) = h_k$ for all k.

This homomorphism also has $\omega(h_k) = e_k$ for all k, so is a self-inverse bijection.

Proof. To show the first claim, it is enough to check that $\mathsf{Sym}_n = \mathbb{Z}[e_1, e_2, \dots, e_n]$ where we truncate to n variables. This is well-known and not too hard to show directly; we will skip the details here.

The main thing left to prove is that $\omega(h_k) = e_k$. This can be shown using generating functions.

Define $H(t) = \sum_{k \geq 0} h_k t^k$ and $E(t) = \sum_{k \geq 0} e_k t^k$.

Argue that $H(t) = \prod_{k \geq 1} (1 - x_k t)^{-1}$ and $E(t) = \prod_{k \geq 1} (1 + x_k t)$, so $H(t)E(-t) = 1$.

Extracting coefficients gives some relations that express the h_k's in terms of the e_k's. But you can observe that these same relations also express the e_k's in terms of the h_k's, so the ring homomorphism $\omega : e_k \mapsto h_k$ must be an involution, as we wanted to show. $\square$

Let $\mathcal{R}_k$ be free abelian group that is spanned by the symbols $[\pi_\lambda^{S_k}]$ as λ ranges over all partitions of k.

This means that the elements of $\mathcal{R}_k$ are formal $\mathbb{Z}$-linear combinations of these symbols.

Given any representation π of S_k, define $[\pi] = \sum_\lambda c_\lambda [\pi_\lambda^{S_k}]$ where $\pi \cong \sum_\lambda (\pi_\lambda^{S_k})^{\oplus c_\lambda}$.

We view the direct sum $\mathcal{R} = \bigoplus_{k\geq 0} \mathcal{R}_k$ as a graded ring by setting

$$[\phi][\psi] := \left[\mathrm{Ind}_{S_k\times S_l}^{S_{k+l}}(\phi\otimes\psi)\right]$$

for representations ϕ of S_k and ψ of S_l. The induced representation is computed by viewing $S_k\times S_l$ as the subgroup of S_{k+l} in which the first factor permutes $1,2,\dots,k$ and the second permutes $k+1,k+2,\dots,k+l$.

The *Frobenius characteristic* $\mathrm{ch} : \mathcal{R} \to \mathsf{Sym}$ is the $\mathbb{Z}$-linear map with $\mathrm{ch}([\pi_\lambda^{S_k}]) = s_\lambda$.

If 1_{S_k} and sgn_{S_k} are the trivial and sign representations of S_k, then $\mathrm{ch}([1_{S_k}]) = h_k$ and $\mathrm{ch}([\mathrm{sgn}_{S_k}]) = e_k$.

Clearly ch is a graded, linear bijection. Our last objective is to show that ch is also a ring isomorphism.

5 See-saws

For this, we talk briefly about *see-saws*. Suppose G_1 and G_2 are groups with subgroups $H_i \subset G_i$.

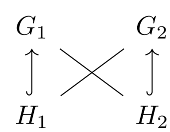
Let Ω be a vector space that is both a G_1-module and a G_2-module. Rather than assuming the actions of G_1 and G_2 commute, we instead assume that the action of G_1 commutes with the action of H_2 and that the action of G_2 commutes with the action of H_1.

This means that we can view Ω as either a $(G_1\times H_2)$-module or a $(G_2\times H_1)$-module.

We say that Ω is a *see-saw* if we have correspondences for both of these actions:

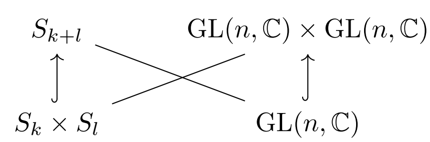
$$\pi_i^{G_1} \xleftrightarrow{\Omega} \sigma_i^{H_2} \qquad \text{and} \qquad \pi_j^{G_2} \xleftrightarrow{\Omega} \sigma_j^{H_1}. \tag{5.1}$$

We indicate this situation with the diagram



(5.2)

Example 5.1. An example of a see-saw is given by



for the vector space $\Omega = (\mathbb{C}^n)^{\otimes(k+l)}$. Here the S_{k+l}-action is the same right action as before.

We view $S_k\times S_l$ as a subgroup of S_{k+l} as discussed earlier. The $\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})$-action on Ω is

$$(g,h)(v_1\otimes\cdots\otimes v_k\otimes v_{k+1}\otimes\cdots\otimes v_{k+l}) = gv_1\otimes\cdots\otimes gv_k\otimes hv_{k+1}\otimes\cdots\otimes hv_{k+l}.$$

Finally, we embed $\mathrm{GL}(n,\mathbb{C})$ as the subgroup $\{(g,g) : g\in\mathrm{GL}(n,\mathbb{C})\}\subset\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})$.

Lemma 5.2. Assume we are in the situation of (5.2) with correspondences (5.1). Then the multiplicity of $\sigma_j^{H_1}$ in the restriction of $\pi_i^{G_1}$ to H_1 is also the multiplicity of $\sigma_i^{H_2}$ in the restriction of $\pi_j^{G_2}$ to H_2.

Proof. Argue that both multiplicities are the dimension of $\mathrm{Hom}_{H_1\times H_2}\left(\sigma_j^{H_1}\otimes\sigma_i^{H_2},\Omega\right)$ since

$$\mathrm{Hom}_{H_1\times H_2}\left(\sigma_j^{H_1}\otimes\sigma_i^{H_2},\Omega\right)\cong\mathrm{Hom}_{H_1}\left(\sigma_j^{H_1},\mathrm{Hom}_{H_2}\left(\sigma_i^{H_2},\Omega\right)\right)\cong\mathrm{Hom}_{H_1}\left(\sigma_j^{H_1},\pi_i^{G_1}\right)$$

and also

$$\mathrm{Hom}_{H_1\times H_2}\left(\sigma_j^{H_1}\otimes\sigma_i^{H_2},\Omega\right)\cong\mathrm{Hom}_{H_2}\left(\sigma_i^{H_2},\mathrm{Hom}_{H_1}\left(\sigma_j^{H_1},\Omega\right)\right)\cong\mathrm{Hom}_{H_2}\left(\sigma_i^{H_2},\pi_j^{G_2}\right).$$

For more details, see the Appendix A.4 in Bump and Schilling's book. □

Theorem 5.3. The Frobenius characteristic map $\mathrm{ch}:\mathcal{R}\to\mathsf{Sym}$ is a ring isomorphism.

Proof. It only remains to show that ch is a multiplicative map.

Let λ and μ be partitions of k and l. Writing $s_\lambda s_\mu=\sum_\nu c_{\lambda\mu}^\nu s_\nu$, it suffices to show that the multiplicity of $\pi_\nu^{S_{k+l}}$ in $\pi_\lambda^{S_k}\otimes\pi_\mu^{S_l}$ induced to S_{k+l} is also the nonnegative integer $c_{\lambda\mu}^\nu$.

By Frobenius reciprocity, the multiplicity we want to calculate is also the multiplicity of $\pi_\lambda^{S_k}\otimes\pi_\mu^{S_l}$ in the representation obtained by restricting $\pi_\nu^{S_{k+l}}$ to $S_k\times S_l$. By the previous proposition applied to the see-saw in Example 5.1, this multiplicity is equal, in turn, to the multiplicity of $\pi_\nu^{\mathrm{GL}(n)}$ in the restriction of $\pi_\lambda^{\mathrm{GL}(n)}\otimes\pi_\mu^{\mathrm{GL}(n)}$ to the subgroup $\{(g,g):g\in\mathrm{GL}(n,\mathbb{C})\}\cong\mathrm{GL}(n,\mathbb{C})$.

This multiplicity is exactly $c_{\lambda\mu}^\nu$, as we need to show, because the relevant characters are the Schur polynomials $s_\nu(t_1,t_2,\ldots,t_n)$ and $s_\lambda(t_1,t_2,\ldots,t_n)s_\mu(t_1,t_2,\ldots,t_n)$ by the Weyl character formula. □