1 Review from last time: Schur-Weyl duality

As usual fix positive integers n and k.

The group $\mathrm{GL}(n,\mathbb{C})$ acts on $(\mathbb{C}^n)^{\otimes k}$ diagonally by $g : v_1 \otimes v_2 \otimes \cdots \otimes v_k \mapsto gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k$.

The symmetric group S_k acts on $(\mathbb{C}^n)^{\otimes k}$ by permuting tensor factors.

These two actions commute with each other. More strongly and less trivially: if we consider the two subrings A and B of the ring of all linear maps $(\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes k}$ generated by the actions of $\mathrm{GL}(n,\mathbb{C})$ and S_k, respectively, then A and B are the *commuting rings* of each other.

This implies $(\mathbb{C}^n)^{\otimes k}$ decomposes into $(\mathrm{GL}(n,\mathbb{C}) \times S_k)$-submodules as $(\mathbb{C}^n)^{\otimes k} = \bigoplus_\lambda V_\lambda^{\mathrm{GL}(n)} \otimes V_\lambda^{S_k}$ where

- λ runs through all partitions of k with at most n parts.
- $V_\lambda^{\mathrm{GL}(n)}$ is an irreducible $\mathrm{GL}(n,\mathbb{C})$-module with $V_\lambda^{\mathrm{GL}(n)} \not\cong V_\mu^{\mathrm{GL}(n)}$ if $\lambda \neq \mu$.
- $V_\lambda^{S_k}$ is an irreducible S_k-module with $V_\lambda^{S_k} \not\cong V_\mu^{S_k}$ if $\lambda \neq \mu$.
- If $n \geq k$ then the $V_\lambda^{S_k}$ represent all non-isomorphic irreducible S_k-modules.

We write $\pi_\lambda^{\mathrm{GL}(n)}$ and $\pi_\lambda^{S_k}$ for the representations corresponding to the modules $V_\lambda^{\mathrm{GL}(n)}$ and $V_\lambda^{S_k}$.

This is consistent with our earlier notation for polynomial $\mathrm{GL}(n,\mathbb{C})$-representations.

The decomposition of $(\mathbb{C}^n)^{\otimes k}$ identifies a correspondence $\pi_\lambda^{\mathrm{GL}(n)} \xleftrightarrow{(\mathbb{C}^n)^{\otimes k}} \pi_\lambda^{S_k}$.

This correspondence is *Schur-Weyl duality*. It gives us a natural embedding of the set of irreducible representations of S_k into the set of irreducible polynomial representations of $\mathrm{GL}(n,\mathbb{C})$ for all $n \geq k$.

This has many generalizations. A similar correspondence exists between the irreducible modules of subrings $A, B \subset \mathrm{End}(\Omega)$ whenever A is the commuting ring of B and B is the commuting ring of A.

2 Extending the RSK correspondence

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with at most n nonzero parts.

The Weyl character formula tells us that the character of the irreducible polynomial representation $\pi_\lambda^{\mathrm{GL}(n)}$ evaluated at a diagonal matrix $t = \mathrm{diag}(t_1, t_2, \dots, t_n)$ is the Schur polynomial $s_\lambda(t_1, t_2, \dots, t_n)$.

Taking $t = 1$ gives $\deg \pi_\lambda^{\mathrm{GL}(n)} = s_\lambda(1, 1, \dots, 1) = |\mathrm{SSYT}_n(\lambda)|$.

It is well-known that $\deg \pi_\lambda^{S_k} = |\mathrm{SYT}(\lambda)|$ where $\mathrm{SYT}(\lambda)$ is the set of *standard tabeaux* of shape λ, i.e., semistandard tableaux of shape λ containing each of the numbers $1, 2, \dots, |\lambda|$ (exactly once, necessarily).

Schur-Weyl duality implies the following enumerative identity:

Corollary 2.1. It holds that

$$n^k = \sum_\lambda |\mathrm{SSYT}_n(\lambda)| \cdot |\mathrm{SYT}(\lambda)|$$

where the sum is over all partitions of k with at most n parts.

Proof. The left-hand side is $\dim (\mathbb{C}^n)^{\otimes k}$ and each summand is the dimension of $V_\lambda^{\mathrm{GL}(n)} \otimes V_\lambda^{S_k}$. □

We have already seen a proof of this result using the RSK correspondence $w \mapsto (P_{\mathsf{RSK}}(w), Q_{\mathsf{RSK}}(w))$.

Recall that if $w = w_1 w_2 \cdots w_k$ then

$$P_{\mathsf{RSK}}(w) = \emptyset \leftarrow w_1 \leftarrow w_2 \leftarrow \cdots \leftarrow w_k$$

where if T is a tableau and $a \in \mathbb{Z}$ then $T \leftarrow a$ is formed as follows:

- At each stage a number x is inserted into a row, starting with a into the first row of T.

  When x is inserted, let y be the first entry in the row with $x < y$.

- If no such entry exists then x is added to the end of the row.

  Otherwise we replace y by x and insert y into the next row.

For example, we have $\begin{array}{cc} 1 & 2 \\ 3 & 3 \end{array} \leftarrow 1 = \begin{array}{cc} 1 & 1 \\ 2 & 3 \\ 3 & \end{array}$.

The tableau $P_{\mathsf{RSK}}(w)$ is always semistandard, and $Q_{\mathsf{RSK}}(w)$ is the standard tableau with the same shape as $P_{\mathsf{RSK}}(w)$ that contains i in the box added by inserting the letter w_i.

The RSK correspondence is a bijection from words with letters in $[n] = \{1, 2, \dots, n\}$ to pairs (P, Q) of tableaux with the same shape in which P is semistandard with entries in $[n]$ and Q is standard.

We begin today by noting a generalization of this bijection.

Suppose $X \in \mathrm{Mat}_{r\times n}(\mathbb{N})$ is an $r \times n$ matrix with nonnegative integer entries.

Form a two-line array $A = \left[\begin{array}{cccc} i_1 & i_2 & \cdots & i_m \\ j_1 & j_2 & \cdots & i_m \end{array}\right]$ from X as follows.

Each column of this array is a pair $\left[\begin{array}{c} i \\ j \end{array}\right]$ such that $X_{ij} \neq 0$. This column is repeated exactly X_{ij} times.

The columns of A are ordered lexicographically, so $i_1 \leq i_2 \leq \cdots \leq i_k$ and if $i_t = i_{t+1}$ then $j_t \leq j_{t+1}$.

For example, $X = \left[\begin{array}{ccc} 1 & 3 & 0 \\ 2 & 0 & 2 \end{array}\right] \rightsquigarrow A = \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 1 & 1 & 3 & 3 \end{array}\right]$.

Now define $P_{\mathsf{RSK}}(X) = P_{\mathsf{RSK}}(j_1 j_2 \cdots j_m)$ and let $Q_{\mathsf{RSK}}(X)$ be the tableau with the same shape of $P_{\mathsf{RSK}}(X)$ that contains i_t in the box added by inserting j_t for each $t = 1, 2, \dots, m$.

For example, if $X = \left[\begin{array}{ccc} 1 & 3 & 0 \\ 2 & 0 & 2 \end{array}\right]$ then $P_{\mathsf{RSK}}(X) = P_{\mathsf{RSK}}(12221133)$ so

$$1 \rightsquigarrow 1\,2 \rightsquigarrow 1\,2\,2 \rightsquigarrow 1\,2\,2\,2 \rightsquigarrow \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & & & \end{array}$$

$$\rightsquigarrow \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & & \end{array}$$

$$\rightsquigarrow \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & & & \end{array}$$

$$\rightsquigarrow \begin{array}{cccccc} 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & & & & \end{array} \rightsquigarrow \begin{array}{cccccc} 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & & & & \end{array} = P_{\mathsf{RSK}}(X).$$

We then have $Q_{\mathsf{RSK}}(12221133) = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & 8 \\ 5 & 6 & & & & \end{array}$ and $Q_{\mathsf{RSK}}(X) = \begin{array}{cccccc} 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & & & & \end{array}$.

Clearly $P_{\mathsf{RSK}}(X)$ is always semistandard.

Lemma 2.2. If $X \in \mathrm{Mat}_{r\times n}(\mathbb{N})$ then $Q_{\mathsf{RSK}}(X)$ is also semistandard.

Proof. It is clear that $Q_{\mathsf{RSK}}(X)$ has weakly increasing rows and columns.

Consider the columns $\left[\begin{array}{c} i \\ j \end{array}\right]$ in the corresponding two-line array A.

When i is fixed, the entries below i are in weakly increasing order.

The boxes added by inserting these entries are the ones labeled i in $Q_{\mathsf{RSK}}(X)$.

It not hard to check that when we insert a sequence of entries in weakly increasing order, the box added by one insertion ends up in a column strictly less than the box added by the next insertion.

Therefore boxes labeled by i in $Q_{\mathsf{RSK}}(X)$ are all in distinct columns.

Thus the columns of $Q_{\mathsf{RSK}}(X)$ are actually strictly increasing, so the tableau in semistandard. □

Suppose Q is a semistandard tableau with m boxes $\Box_1, \Box_2, \dots, \Box_m$.

Assume these boxes are indexed in the unique way such that if $i < j$ then either the entry of $\Box_i$ is less than that of $\Box_j$, or the entries in the boxes are equal but the column of $\Box_i$ is less than the column of $\Box_j$.

Define the *standardization* of Q to be the tableau $\mathsf{std}(Q)$ formed by replacing $\Box_i$ by i for all $i \in [m]$.

Since Q is semistandard, $\mathsf{std}(Q)$ is standard. For example,

$$\mathsf{std}\left(\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}\right) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 7 & 8 \\ \hline 5 & 6 \\ \cline{1-2} \end{array}.$$

As we see in this example, if X corresponds to the two-line array $A = \left[\begin{array}{cccc} i_1 & i_2 & \cdots & i_m \\ j_1 & j_2 & \cdots & i_m \end{array}\right]$ then

$$Q_{\mathsf{RSK}}(i_1 i_2 \cdots i_m) = \mathsf{std}(Q_{\mathsf{RSK}}(X))$$

while by definition $P_{\mathsf{RSK}}(j_1 j_2 \cdots j_m) = P_{\mathsf{RSK}}(X)$.

Theorem 2.3. Let r and n be positive integers. Then the map $X \mapsto (P_{\mathsf{RSK}}(X), Q_{\mathsf{RSK}}(X))$ is a bijection

$$\mathrm{Mat}_{r\times n}(\mathbb{N}) \to \bigsqcup_{\lambda} \mathrm{SSYT}_n(\lambda) \times \mathrm{SSYT}_r(\lambda)$$

of all $r \times n$ matrices with nonnegative integer entries to the set of pairs (P, Q) of semistandard tableaux of the same shape, where P has entries in $[n]$ and Q has entries in $[r]$.

Proof. It suffices to construct an inverse map.

We leverage the fact that we already know how to invert the RSK correspondence for words.

Given $(P, Q) = (P_{\mathsf{RSK}}(X), Q_{\mathsf{RSK}}(X))$, define $j_1 j_2 \cdots j_m$ to be word obtained by obtained the inverse RSK correspondence to the pair $(P, \mathsf{std}(Q))$, in which the second tableau in standard.

Then let $i_1 i_2 \cdots i_m$ be the sequence where i_t is then entry in the box of Q that contains t in $\mathsf{std}(Q)$.

We then recover X as the unique matrix corresponding to the array $A = \left[\begin{array}{cccc} i_1 & i_2 & \cdots & i_m \\ j_1 & j_2 & \cdots & i_m \end{array}\right]$.

This correspondence $(P, Q) \mapsto X$ is the map described in theorem, so both maps are bijections. □

There is a followup result we should mention, which is proved in Chapter 7 of Bump and Schilling's book.

Fix a nonnegative integer matrix $X \in \mathrm{Mat}_{r\times n}(\mathbb{N})$.

Theorem 2.4. If $(P, Q) = (P_{\mathsf{RSK}}(X), Q_{\mathsf{RSK}}(X))$, then $(Q, P) = (P_{\mathsf{RSK}}(X^T), Q_{\mathsf{RSK}}(X^T))$.

3 The $\mathrm{GL}(n) \times \mathrm{GL}(r)$ bicrystal

Let n and r be positive integers.

Consider partitions λ and μ with at most n and at most r parts, respectively.

The sets $\mathrm{SSYT}_n(\lambda)$ is a $\mathrm{GL}(n)$-crystal while $\mathrm{SSYT}_r(\mu)$ is a $\mathrm{GL}(r)$-crystal.

Thus $\mathrm{SSYT}_n(\lambda) \times \mathrm{SSYT}_r(\lambda)$ is a crystal for the product Cartan type $\mathrm{GL}(n) \times \mathrm{GL}(r)$.

Observation 3.1. The set $\mathrm{Mat}_{r\times n}(\mathbb{N})$ of $r \times n$ matrices with nonnegative integer entries has a unique $\mathrm{GL}(n)\times\mathrm{GL}(r)$ crystal structure such that the generalized RSK correspondence $X \mapsto (P_{\mathsf{RSK}}(X), Q_{\mathsf{RSK}}(X))$ is a crystal isomorphism $\mathrm{Mat}_{r\times n}(\mathbb{N}) \cong \bigsqcup_\lambda \mathrm{SSYT}_n(\lambda) \times \mathrm{SSYT}_r(\lambda)$.

We view $\mathrm{Mat}_{r\times n}(\mathbb{N})$ as a crystal in this way.

It is pleasantly simple to implement this crystal structure on nonnegative integer matrices directly.

The $\mathrm{GL}(n)$- and $\mathrm{GL}(r)$-crystal structure will correspond to operations on rows and columns, respectively.

Let $\Lambda = \mathbb{Z}^n$ and $\Lambda' = \mathbb{Z}^r$. Fix a matrix $X \in \mathrm{Mat}_{r\times n}(\mathbb{N})$.

The $\mathrm{GL}(n)$ weight map is given by the column sums $\mathbf{wt}(X) = (s_1, s_2, \dots, s_n)$ where $s_j := \sum_{i=1}^r X_{ij}$.

The $\mathrm{GL}(r)$ weight map is given by the row sums $\mathbf{wt}'(X) = (s'_1, s'_2, \dots, s'_r)$ where $s'_i := \sum_{j=1}^n X_{ij}$.

To define the $\mathrm{GL}(n)$ crystal operators, let

$$\begin{aligned}\Psi_i(X,k) &= (a_1 + a_2 + \dots + a_k) - (b_1 + b_2 + \dots + b_{k-1})\\ \Delta_i(X,k) &= (b_k + b_{k+1} + \dots + b_r) - (a_{k+1} + a_{k+2} + \dots + a_r)\end{aligned}$$

where columns i and $i+1$ of X are

$$\begin{bmatrix} X_{1,i} & X_{1,i+1} \\ X_{2,i} & X_{2,i+1} \\ \vdots & \vdots \\ X_{k,i} & X_{k,i+1} \\ X_{k+1,i} & X_{k+1,i+1} \\ \vdots & \vdots \\ X_{r,i} & X_{r,i+1} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_k & b_k \\ a_{k+1} & b_{k+1} \\ \vdots & \vdots \\ a_r & b_r \end{bmatrix}.$$

Write $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1} = (0,\dots,0,1,-1,0,\dots,0) \in \mathbb{Z}^n$, viewed as a row vector.

Now define $\varphi_i(X) = \max_{1\le k\le r} \Psi_i(X,k)$ and $\varepsilon_i(X) = \max_{1\le k\le r} \Delta_i(X,k)$.

If $\varphi_i(X) = 0$ then $f_i(X) = 0$. If $\varphi_i(X) \neq 0$ then $f_i(X)$ is obtained by subtracting α_i to the kth row of X, where k is the first value where $\Psi_i(X,k)$ attains its maximum.

Likewise, if $\varepsilon_i(X) = 0$ then $e_i(X) = 0$. If $\varepsilon_i(X) \neq 0$ then $e_i(X)$ is obtained by adding α_i to the kth row of X, where k is the last value where $\Delta_i(X,k)$ attains its maximum.

The $\mathrm{GL}(r)$ crystal operators are defined by transposing everything in sight.

For example, we set $\varphi'_i(X) = \varphi_i(X^T)$ and $\varepsilon'_i(X) = \varepsilon_i(X^T)$.

If $\varphi'_i(X) = 0$ then $f'_i(X) = 0$ and if $\varepsilon'_i(X) = 0$ then $e'_i(X) = 0$.

Otherwise, $f'_i(X) = f_i(X^T)^T$ and $e'_i(X) = e_i(X^T)^T$.

These operators can be defined in terms of adding or subtracting the column vector $\alpha'_i = \mathbf{e}_i - \mathbf{e}_{i+1} \in \mathbb{Z}^r$ from the kth column of X, where k is the first or last value at which the quantities

$$\Psi'_i(X,k) := \Psi_i(X^T,k) \qquad \text{or} \qquad \Delta'_i(X,k) := \Delta_i(X^T,k)$$

attains their maximum.

Theorem 3.2. The $\mathrm{GL}(n)$-crystal structure on $\mathrm{Mat}_{r\times n}(\mathbb{N})$ corresponds to the weight map $\mathbf{wt}$, string lengths φ_i and ε_i for $i \in [n-1]$, and crystal operators e_i and f_i for $i \in [n-1]$ just described.

The $\mathrm{GL}(r)$-crystal structure on $\mathrm{Mat}_{r\times n}(\mathbb{N})$ corresponds to the weight map $\mathbf{wt}'$, string lengths φ'_i and ε'_i for $i \in [r-1]$, and crystal operators e'_i and f'_i for $i \in [r-1]$ just described.

These structures are compatible and correspond to the $\mathrm{GL}(n)\times\mathrm{GL}(r)$-crystal structure on $\mathrm{Mat}_{r\times n}(\mathbb{N})$.

In particular, for any matrix $X \in \mathrm{Mat}_{r\times n}(\mathbb{N})$ we have

$$\begin{aligned} P_{\mathsf{RSK}}(e_i(X)) = e_i(P_{\mathsf{RSK}}(X)) \qquad &\text{and} \qquad Q_{\mathsf{RSK}}(e_i(X)) = Q_{\mathsf{RSK}}(X), \\ P_{\mathsf{RSK}}(f_i(X)) = f_i(P_{\mathsf{RSK}}(X)) \qquad &\text{and} \qquad Q_{\mathsf{RSK}}(f_i(X)) = Q_{\mathsf{RSK}}(X), \end{aligned}$$

for each $i \in [n-1]$ with $f_i(X) \neq 0$ or $e_i(X) \neq 0$ as appropriate, and

$$\begin{aligned} P_{\mathsf{RSK}}(e'_i(X)) = P_{\mathsf{RSK}}(X) \qquad &\text{and} \qquad Q_{\mathsf{RSK}}(e'_i(X)) = e_i(Q_{\mathsf{RSK}}(X)), \\ P_{\mathsf{RSK}}(f'_i(X)) = P_{\mathsf{RSK}}(X) \qquad &\text{and} \qquad Q_{\mathsf{RSK}}(f'_i(X)) = f_i(Q_{\mathsf{RSK}}(X)), \end{aligned}$$

for each $i \in [r-1]$ with $f'_i(X) \neq 0$ or $e'_i(X) \neq 0$ as appropriate.

Proof. The details are a little technical but straightforward.

Because of Theorem 2.4, it is only necessary to check these assertions for the $\mathrm{GL}(n)$-crystal structure.

One can show that the string lengths and crystal operators act as we expect by matching up some general formulas for iterated tensor products, using the fact the P_{RSK} is constant on plactic equivalence classes.

The claim that $Q_{\mathsf{RSK}}(e_i(X)) = Q_{\mathsf{RSK}}(X)$ and $Q_{\mathsf{RSK}}(f_i(X)) = Q_{\mathsf{RSK}}(X)$ is deduced by tracing through the definitions and using the fact that words belong to the same connected component of $\mathbb{B}_n^{\otimes k}$ if and only if have they have the same image under Q_{RSK}.

The full arguments are in Chapter 9 of Bump and Schilling's book. □

4 The crystal see-saw and the Littlewood-Richardson rule

Let n, r, and s be positive integers.

We have just described a $\mathrm{GL}(n)\times\mathrm{GL}(r+s)$ crystal structures on $\mathrm{Mat}_{(r+s)\times n}(\mathbb{N})$.

Since can write any matrix $X \in \mathrm{Mat}_{(r+s)\times n}(\mathbb{N})$ as two stacked matrices

$$X = \begin{bmatrix} X' \\ X'' \end{bmatrix} \qquad \text{for } X' \in \mathrm{Mat}_{r\times n}(\mathbb{N}) \text{ and } X'' \in \mathrm{Mat}_{s\times n}(\mathbb{N}),$$

the set $\mathrm{Mat}_{(r+s)\times n}(\mathbb{N})$ also has a $\mathrm{GL}(n)\times\mathrm{GL}(r)\times\mathrm{GL}(n)\times\mathrm{GL}(s)$ crystal structure.

Lemma 4.1. In the above situation, let (P,Q), (P',Q'), and (P'',Q'') be the results of applying RSK to the matrices X, X', and X'', respectively. Then P is plactically equivalent to $P'\otimes P''$ as elements of $\mathrm{GL}(n)$ crystals, while Q is plactically equivalent to (Q',Q'') as elements of $\mathrm{GL}(r)\times\mathrm{GL}(s)$ crystals.

Proof. Suppose the two-line array associated to X is

$$A = \left[\begin{array}{cccc} i_1 & i_2 & \cdots & i_m \\ j_1 & j_2 & \cdots & i_m \end{array} \right].$$

Then P is plactically equivalently to the word $j_1 j_2 \cdots j_m \in \mathbb{B}_n^{\otimes m}$ while P' and P' are plactically equivalent to $j_1 j_2 \cdots j_k \in \mathbb{B}_n^{\otimes k}$ and $j_{k+1} \cdots j_m \in \mathbb{B}_n^{\otimes (m-k)}$, respectively, where k is the last index with $i_k \leq r$.

Hence $P' \otimes P'' \equiv j_1 j_2 \cdots j_k \otimes j_{k+1} \cdots j_m = j_1 j_2 \cdots j_m \equiv P$.

Other the other hand, let C_j be the weakly increasing word formed by repeating the row index i of each nonzero entry in column j of X eactly X_{ij} times. In view of Theorem 2.4, Q is $\mathrm{GL}(r+s)$-plactically equivalent to $C_1 C_2 \cdots C_n = C_1 \otimes C_2 \otimes \cdots \otimes C_n$.

Each C_j can be written as $C_j = C_j' C_j''$ where C_j' are has all letters in $\{1, \ldots, r\}$ and C_j'' has all letters in $\{r+1, \ldots, s\}$. As an element of a (branched) $\mathrm{GL}(r) \times \mathrm{GL}(s)$-crystal, Q is plactically equivalent to

$$(C_1', C_1'') \otimes (C_2', C_2'') \otimes \cdots \otimes (C_n', C_n'') \equiv (C_1' C_2' \cdots C_n', C_1'' C_2'' \cdots C_n'').$$

Again using Theorem 2.4, we have $Q' \equiv C_1' C_2' \cdots C_n'$ as elements of $\mathrm{GL}(r)$ crystals and $Q'' \equiv C_1'' C_2'' \cdots C_n''$ as elements of $\mathrm{GL}(s)$ crystals, so $Q \equiv (Q', Q'')$. □

Theorem 4.2. Let λ, μ, and ν be partitions.

Then the multiplicity of $\mathrm{SSYT}_n(\lambda)$ in $\mathrm{SSYT}_n(\mu) \otimes \mathrm{SSYT}_n(\nu)$ equals the multiplicity of $\mathrm{SSYT}_r(\mu) \times \mathrm{SSYT}_s(\nu)$ in the $\mathrm{GL}(r) \times \mathrm{GL}(s)$ crystal obtained by branching $\mathrm{SSYT}_{r+s}(\lambda)$.

In other words, our two interpretations of the Littlewood-Richardson coefficients $c_{\mu\nu}^{\lambda}$, as either the number of skew tableaux of shape λ/μ with weight ν whose reading words are Yamanouchi words, or as the multiplicity of s_λ in the product of Schur functions $s_\mu s_\nu$, are consistent.

Proof. Consider the set

$$\mathcal{C} := \{X \in \mathrm{Mat}_{(r+s)\times n}(\mathbb{N}) : P_{\mathsf{RSK}}(X) \in \mathrm{SSYT}_n(\lambda),\ Q_{\mathsf{RSK}}(X') \in \mathrm{SSYT}_r(\mu),\ Q_{\mathsf{RSK}}(X'') \in \mathrm{SSYT}_s(\nu)\}$$

which consists of all elements of $\mathrm{Mat}_{(r+s)\times n}(\mathbb{N})$ that are $(\mathrm{GL}(n) \times \mathrm{GL}(r) \times \mathrm{GL}(s))$-plactically equivalent to elements of the $\mathrm{GL}(n) \times \mathrm{GL}(r) \times \mathrm{GL}(s)$ crystal $\mathrm{SSYT}_n(\lambda) \times \mathrm{SSYT}_r(\mu) \times \mathrm{SSYT}_s(\nu)$. The set $\mathcal{C}$ is a disjoint union of copies of this crystal and the idea is to count these copies in two different ways.

On other hand, we have $\mathcal{C} \subset \{X \in \mathrm{Mat}_{(r+s)\times n}(\mathbb{N}) : P_{\mathsf{RSK}}(X) \in \mathrm{SSYT}_n(\lambda)\}$. This is a $\mathrm{GL}(n) \times \mathrm{GL}(r+s)$ crystal isomorphic to $\mathrm{SSYT}_n(\lambda) \times \mathrm{SSYT}_{r+s}(\lambda)$. On branching to $\mathrm{GL}(n) \times \mathrm{GL}(r) \times \mathrm{GL}(s)$, the number of subcrystals isomorphic to $\mathrm{SSYT}_n(\lambda) \times \mathrm{SSYT}_r(\mu) \times \mathrm{SSYT}_s(\nu)$ is equal to the multiplicity of $\mathrm{SSYT}_r(\mu) \times \mathrm{SSYT}_\nu(s)$ in the $\mathrm{GL}(r) \times \mathrm{GL}(s)$ crystal obtained by branching $\mathrm{SSYT}_{r+s}(\lambda)$.

On the other hand, $\mathcal{C} \subset \{X \in \mathrm{Mat}_{(r+s)\times n}(\mathbb{N}) : Q_{\mathsf{RSK}}(X') \in \mathrm{SSYT}_r(\mu),\ Q_{\mathsf{RSK}}(X'') \in \mathrm{SSYT}_s(\nu)\}$. This is isomorphic to $\mathrm{SSYT}_n(\mu) \times \mathrm{SSYT}_n(\nu) \times \mathrm{SSYT}_r(\mu) \times \mathrm{SSYT}_s(\nu)$ as a $\mathrm{GL}(n) \times \mathrm{GL}(n) \times \mathrm{GL}(r) \times \mathrm{GL}(s)$ crystal. Since by the lemma $P_{\mathsf{RSK}}(X) \equiv P_{\mathsf{RSK}}(X') \otimes P_{\mathsf{RSK}}(X'')$, it follows that the number of subcrystals isomorphic to $\mathrm{SSYT}_n(\lambda) \times \mathrm{SSYT}_r(\mu) \times \mathrm{SSYT}_s(\nu)$ equals the multiplicity of $\mathrm{SSYT}_n(\lambda)$ in $\mathrm{SSYT}_n(\mu) \otimes \mathrm{SSYT}_n(\nu)$.

The theorem follows by comparing these two multiplicity calculations. □