1 Last time: the $\mathrm{GL}(n) \times \mathrm{GL}(r)$ bicrystal

Last time, we extended the domain of the RSK correspondence from words to matrices $X \in \mathrm{Mat}_{r\times n}(\mathbb{N})$.

The idea is to first turn a nonnegative integer matrix X into a two-line array $\begin{bmatrix} i_1 & i_2 & \cdots & i_m \\ j_1 & j_2 & \cdots & i_m \end{bmatrix}$.

Each column in this array is a pair $\begin{bmatrix} i \\ j \end{bmatrix}$ such that $X_{ij} \neq 0$. This column is repeated exactly X_{ij} times.

The columns are ordered lexicographically.

Then $P_{\mathsf{RSK}}(X) = P_{\mathsf{RSK}}(j_1 j_2 \cdots j_m)$ and we form $Q_{\mathsf{RSK}}(X)$ by replacing each k in $Q_{\mathsf{RSK}}(j_1 j_2 \cdots j_m)$ by i_k.

Key fact: $P_{\mathsf{RSK}}(X^T) = Q_{\mathsf{RSK}}(X)$ and $Q_{\mathsf{RSK}}(X^T) = P_{\mathsf{RSK}}(X)$.

The map $X \mapsto (P_{\mathsf{RSK}}(X), Q_{\mathsf{RSK}}(X))$ is a bijection $\mathrm{Mat}_{r\times n}(\mathbb{N}) \xrightarrow{\sim} \bigsqcup_\lambda \mathrm{SSYT}_n(\lambda) \times \mathrm{SSYT}_r(\lambda)$ where the union is over all partitions with at most $\min\{r, n\}$ parts. We give $\mathrm{Mat}_{r\times n}(\mathbb{N})$ the unique $\mathrm{GL}(n) \times \mathrm{GL}(r)$ crystal structure that makes this bijection into a crystal isomorphism.

One can describe the $\mathrm{GL}(n) \times \mathrm{GL}(r)$ crystal on $\mathrm{Mat}_{r\times n}(\mathbb{N})$ directly, without reference to RSK.

As an application, we showed that the Littlewood-Richardson coefficient $c^\lambda_{\mu\nu}$ is both the multiplicity of $\mathrm{SSYT}_n(\lambda)$ in $\mathrm{SSYT}_n(\mu) \otimes \mathrm{SSYT}_n(\nu)$ and the multiplicity of $\mathrm{SSYT}_r(\mu) \times \mathrm{SSYT}_s(\nu)$ in the $\mathrm{GL}(r) \times \mathrm{GL}(s)$ crystal obtained by branching $\mathrm{SSYT}_{r+s}(\lambda)$.

2 The Cauchy correspondence

We turn to Appendix B of Bump and Schilling's book, which covers some results from representation theory that are analogous to our results on $\mathrm{GL}(n) \times \mathrm{GL}(r)$ bicrystals from last time.

An *affine algebraic group* Γ is an affine algebraic variety that is also a group such the the group multiplication map $\Gamma \times \Gamma \to \Gamma$ and the inverse map $\Gamma \to \Gamma$ are regular.

Our main example is the affine algebraic group $\Gamma = \mathrm{GL}(n, \mathbb{C})$.

Let $A = \mathcal{O}(\Gamma)$ be the coordinate ring of regular functions on Γ. This is a finitely generated $\mathbb{C}$-algebra that is a reduced Noetherian ring. Since $\mathcal{O}(\Gamma \times \Gamma) = A \otimes A$, the multiplication map $\nabla : \Gamma \times \Gamma \to \Gamma$ corresponds to an algebra homomorphism $\Delta : A \to A \otimes A$ with the formula $\Delta(f) := f \circ \nabla$.

This *comultiplication* map $\Delta : A \to A \otimes A$ makes the ring A into a commutative Hopf algebra, whose antipode is the map $S(f) := \big(x \mapsto f(x^{-1})\big)$.

As usual, a *finite-dimensional representation* of Γ means a pair (π, V) where V is a finite-dimensional complex vector space and $\pi : \Gamma \to \mathrm{GL}(V)$ is a regular map. When $\Gamma = \mathrm{GL}(n, \mathbb{C})$, this reduces to our earlier definition.

We assume that every finite-dimensional representation decomposes into a direct sum of irreducible subrepresentations. In the case when $\Gamma = \mathrm{GL}(n, \mathbb{C})$ this follows from Weyl's unitarian trick.

The product group $\Gamma \times \Gamma$ acts on A by $(g_1, g_2) \cdot f = \big(x \mapsto f\left(g_2^{-1} x g_1\right)\big)$.

This is indeed an action: $(g_3, g_4) \cdot ((g_1, g_2) \cdot f)$ for $g_i \in \Gamma$ and $f \in A$ is the map

$$x \mapsto ((g_1, g_2) \cdot f)\left(g_4^{-1} x g_3\right) = f\left(g_2^{-1} g_4^{-1} x g_3 g_1\right),$$

which is also the formula for $(g_3 g_1, g_4 g_2) \cdot f$.

If (π, V) is a finite-dimensional representation then let $\hat{\pi} \in \mathrm{GL}(V^*)$ be the map

$$\hat{\pi}(g) : \lambda \mapsto \lambda \circ \pi(g^{-1}).$$

The pair $(\hat{\pi}, V^*)$ is another finite-dimensional representation, called the *contragredient representation*.

Proposition 2.1. The coordinate ring $A = \mathcal{O}(\Gamma)$ decomposes as a $\Gamma \times \Gamma$ module as

$$A \cong \bigoplus \pi \otimes \hat{\pi}$$

where the summation runs over all isomorphism classes of finite-dimensional representations π of Γ.

Proof sketch. Let (π, V) be a finite-dimensional representation.

Define an embeding $\iota : V \otimes V^* \to A$ by mapping $v \otimes \lambda$, where $v \in V$ and $\lambda \in V^*$, to the function

$$\iota(v \otimes \lambda) : g \mapsto \lambda\left(\pi(g)v\right).$$

Note that $\Gamma \times \Gamma$ acts on $V \otimes V^*$ by $(g_1, g_2) \cdot (v \otimes \lambda) = \pi(g_1)v \otimes \lambda \circ \pi(g_2^{-1})$. Thus

$$\iota\left((g_1, g_2) \cdot (v \otimes \lambda)\right) : x \mapsto \lambda\left(\pi(g_2^{-1})\pi(x)\pi(g_1)v\right) = \iota(v \otimes \lambda)(g_2^{-1}xg_1) = \left((g_1, g_2) \cdot \iota(v \otimes \lambda)\right)(x).$$

In other words $\iota\left((g_1, g_2) \cdot (v \otimes \lambda)\right) = (g_1, g_2) \cdot \iota(v \otimes \lambda)$.

Thus the map $\iota : V \otimes V^* \to A$ is a $\Gamma \times \Gamma$-module homomorphism.

One needs to show that every element $f \in A$ is a linear combination of functions of the form $\iota(v \otimes \lambda)$.

Fix $f \in A$ and write $\Delta f = \sum_i \phi_i \otimes \psi_i$ where $\phi_i, \psi_i \in A$, so that if $x, y \in \Gamma$ then $f(xy) = \sum_i \phi_i(x)\psi_i(y)$.

Take V to be the vector space spanned by the right-translations $(y, 1) \cdot f$ for $y \in \Gamma$.

This vector space space is finite-dimensional since it is spanned by the finite set of functions ϕ_i that appear in the formula $\Delta f = \sum_i \phi_i \otimes \psi_i$. Specifically, if $y \in \Gamma$ then $(y, 1) \cdot f = \sum_i \psi_i(y)\phi_i$ since

$$((y, 1) \cdot f)(x) = f(xy) = \sum_i \phi_i(x)\psi_i(y).$$

Now suppose $\pi : \Gamma \to \mathrm{GL}(V)$ is the action by right translation, so that $\pi(g)\phi(x) = \phi(xg)$ for $\phi \in V$.

Finally let $\lambda : V \to \mathbb{C}$ be the map $\lambda(\phi) = \phi(1)$.

Then we have $f = \iota(f \otimes \lambda)$ since $\iota(f \otimes \lambda)(x) = \lambda(\pi(x)f) = (\pi(x)f)(1) = f(x)$.

The result follows using our assumption that every representation is a direct sum of irreducibles. □

Lemma 2.2. If π is a finite-dimensional representation of $\mathrm{GL}(n, \mathbb{C})$, then $\hat{\pi}$ is isomorphic to the representation $g \mapsto \pi\left((g^{-1})^T\right)$, where g^T is the usual transpose of g.

Proof. You can check that if $t_1, t_2, \ldots, t_n$ are the eigenvalues of $\pi(g)$ then $\chi_{\hat{\pi}}(g) = \sum_{i=1}^n t_i^{-1}$. Each $g \in \mathrm{GL}(n, \mathbb{C})$ is conjugate to its transpose (one strategy to prove this: reduce to the case of a single block in the Jordan canonical form).

This means the character of π evaluated at $(g^{-1})^T$ is the same as the value at g^{-1}. Conclude that the representations $g \mapsto \pi\left((g^{-1})^T\right)$ and $\hat{\pi}$ have the same character, so they are isomorphic. □

The *tensor algebra* $T(V)$ of a vector space V is the algebra generated by all tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ where $v_1, v_2, \ldots, v_k \in V$ and $k \geq 0$. The product for this algebra is just $\otimes$. The *symmetric algebra* $\bigvee V$ on V is the quotient of $T(V)$ by the two-sided ideal generated by all differences $x \otimes y - y \otimes x$.

Thus, $\bigvee V$ is the space of tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ where we are allowed to commute the tensor factors.

If V is a finite dimensional complex vector space then $\bigvee V \cong \mathbb{C}[x_1, x_2, \ldots, x_n]$ for $n = \dim V$.

If G_1 is a group acting on V and G_2 is a group acting on W, then $G_1 \times G_2$ acts on $V \otimes W$ and therefore on the symmetric algebra $\bigvee(V \otimes W)$.

Lemma 2.3. There is a decomposition of graded $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$-representations

$$\bigvee (\mathbb{C}^n \otimes \mathbb{C}^n) \cong \bigoplus_\lambda \pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\lambda^{\mathrm{GL}(n)}$$

where λ runs through all partitions of with at most n nonzero parts.

The grading on the right is such that elements of $\pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\lambda^{\mathrm{GL}(n)}$ are homogeneous of degree $|\lambda|$.

Proof sketch. Let $G = \mathrm{GL}(n,\mathbb{C})$.

Instead of the given action, consider the action on the dual space $\mathrm{Mat}_{n\times n}(\mathbb{C})^*$ by

$$(g_1, g_2)f := \big(X \mapsto f(g_2^T X g_1)\big) \qquad \text{for } g_1, g_2 \in G \text{ and } f \in \mathrm{Mat}_{n\times n}(\mathbb{C})^*. \tag{*}$$

This action is equivalent to the action on $\mathbb{C}^n \otimes \mathbb{C}^n$ and the symmetric algebra over $\mathrm{Mat}_{n\times n}(\mathbb{C})^*$ is the same as the affine algebra $\mathcal{O}(\mathrm{Mat}_{n\times n}(\mathbb{C}))$, which is the polynomial ring generated by the coordinate functions g_{ij} for a matrix $g = (g_{ij})$.

To understand this ring, we first discuss the decomposition of the affine algebra $\mathcal{O}(\mathrm{GL}(n,\mathbb{C})) = \mathbb{C}[g_{ij}, \det^{-1}]$.

We have two slightly different actions of $G \times G$ on this algebra.

For the action $(g_1, g_2)f : X \mapsto f(g_2^{-1} X g_1)$, we have $\mathcal{O}(\mathrm{GL}(n,\mathbb{C})) = \bigoplus_\lambda \pi_\lambda^{\mathrm{GL}(n)} \otimes \hat{\pi}_\lambda^{\mathrm{GL}(n)}$.

For the action (*), Lemma 2.2 implies that we instead have $\mathcal{O}(\mathrm{GL}(n,\mathbb{C})) = \bigoplus_\lambda \pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\lambda^{\mathrm{GL}(n)}$.

In both decompositions, the direct sums are over all dominant weights $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$, which are only partitions if $\lambda_n > 0$.

We are interested in the decomposition of the affine algebra $\mathrm{Mat}_{n\times n}(\mathbb{C})$. Since $G = \mathrm{GL}(n,\mathbb{C})$ is an open subvariety of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ (namely, the subset where the determinant is nonzero), the restriction map $\bigvee \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathcal{O}(\mathrm{GL}(n,\mathbb{C}))$ is injective. The representations π_λ that extend to regular functions on $\mathrm{Mat}_{n\times n}(\mathbb{C})$ are precisely those that do not involve inverse powers of the determinant, namely, those that are indexed by dominant weights λ that are partitions. This implies the lemma. □

The theorem implies a related identity for symmetric functions:

Theorem 2.4 (Cauchy identity)**.** Suppose $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ are sequences of commuting indeterminates that commute with each other and with the indeterminate t. Then

$$\prod_{i,j\geq 1} (1 - x_i y_j t)^{-1} = \sum_\lambda s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) t^{|\lambda|}$$

where λ runs through all partitions. Here, the factors on the left are interpreted as rational power series

$$(1 - x_i y_j t)^{-1} = 1 + x_i y_j t + (x_i y_j)^2 t^2 + (x_i y_j)^3 t^3 + \dots.$$

Proof. Suppose $g_1 \in \mathrm{GL}(n,\mathbb{C})$ has distinct eigenvalues $x_1, x_2, \dots, x_n$ and $g_2 \in \mathrm{GL}(n,\mathbb{C})$ has distinct eigenvalues $y_1, y_2, \dots, y_n$, then the trace of the action of (g_1, g_2) on $V = \mathbb{C}^n \otimes \mathbb{C}^n$ is the sum $\sum_{i=1}^n \sum_{j=1}^n x_i y_j$ which is the coefficient of t in $\prod_{i=1}^n \prod_{j=1}^n (1 - \lambda_i y_j t)^{-1}$.

Similarly, the trace of the action of (g_1, g_2) the subspace of homogeneous elements of $\bigvee V$ of degree k is the coefficient of t^k in $\prod_{i=1}^n \prod_{j=1}^n (1 - x_i y_j t)^{-1}$. It follows that $\prod_{i=1}^n \prod_{j=1}^n (1 - x_i y_j t)^{-1}$ is the value of graded character of the $G_1 \times G_2$-representation on $\bigvee V$ evaluated at (g_1, g_2).

On other hand, the Weyl character formula tells us that that graded character of $\bigoplus_\lambda \pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\lambda^{\mathrm{GL}(n)}$ evaluated at (g_1, g_2) is $\sum_\lambda s_\lambda(x_1, x_2, \dots, x_n) s_\lambda(y_1, y_2, \dots, y_n) t^{|\lambda|}$. Since this is the same as the graded character of $\bigvee V$ by the theorem, we get a finite version of the Cauchy identity, in terms of Schur polynomials rather than Schur functions. Letting $n \to \infty$ transforms this to the result. □

Theorem 2.5 (Cauchy correspondence)**.** Let $G_1 = \mathrm{GL}(n, \mathbb{C})$ and $G_2 = \mathrm{GL}(m, \mathbb{C})$ where $m,\, n > 0$.

There is a decomposition of graded $G_1 \times G_2$-representations

$$\bigvee (\mathbb{C}^n \otimes \mathbb{C}^m) \cong \bigoplus_{\lambda} \pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\lambda^{\mathrm{GL}(m)}$$

where λ runs through all partitions of with at most $\min\{m, n\}$ nonzero parts.

The grading on the right is such that elements of $\pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\lambda^{\mathrm{GL}(m)}$ are homogeneous of degree $|\lambda|$.

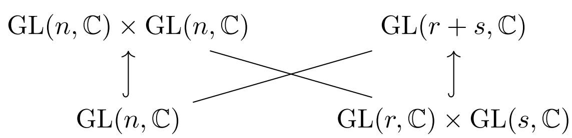
Proof. It is enough to check that the graded characters of both sides coincide. This follows by setting the variables $x_{n+1} = x_{n+2} = \cdots = 0$ and $y_{m+1} = y_{m+2} = \cdots = 0$ in the Cauchy identity. □

The representation $\Omega = \bigvee(\mathbb{C}^n \otimes \mathbb{C}^m)$ gives us a *correspondence*

$$\pi_\lambda^{\mathrm{GL}(n)} \xleftrightarrow{\ \Omega\ } \pi_\lambda^{\mathrm{GL}(m)}$$

in the sense of Lecture 12. This is called $\mathrm{GL}(n) \times \mathrm{GL}(m)$-*duality* or the *Cauchy correspondence.*

We also have a *see-saw*



Here, $\mathrm{GL}(n, \mathbb{C})$ is viewed as the subgroup $\{(g, g) : g \in \mathrm{GL}(n, \mathbb{C})\} \subset \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$.

Likewise, $\mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(s, \mathbb{C})$ is viewed as the subgroup of relevant block diagonal matrices

$$\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} \in \mathrm{GL}(r + s, \mathbb{C}).$$

One diagonal line in this see-saw is the Cauchy correspondence for $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(r + 1, \mathbb{C})$. The other consists of the Cauchy correspondences for $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(r, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(s, \mathbb{C})$ tensored together.

Corollary 2.6. Let λ, μ, and ν be partitions with $|\lambda| = |\mu| + |\nu|$.

Assume $n \geq |\lambda|$, $r \geq |\mu|$, and $s \geq |\nu|$.

Then the multiplicity of $\pi_\lambda^{\mathrm{GL}(n)}$ in $\pi_\mu^{\mathrm{GL}(n)} \times \pi_\nu^{\mathrm{GL}(n)}$ is the same as the multiplicity of $\pi_\mu^{\mathrm{GL}(r)} \otimes \pi_\nu^{\mathrm{GL}(s)}$ in the restriction of $\pi_\lambda^{\mathrm{GL}(r+s)}$ to $\mathrm{GL}(r, \mathbb{C}) \times \mathrm{GL}(s, \mathbb{C})$.

Both numbers are equal to the Littlewood-Richardson coefficient $c_{\mu\nu}^\lambda$.

Proof. This follows by the general properties of see-saws from Lecture 12 applied to our specific case. □

The last result in Lecture 13 (see the end of today's Section 1) is a crystal analogue of this corollary.

3 Crystals for Stanley symmetric functions

In the second half of today's lecture we give a brief survey of Chapter 10 of Bump and Schilling's book.

We construct S_k as the group of bijections $[k] \to [k] := \{1, 2, \ldots, k\}$.

Let $s_i = (i, i + 1) \in S_k$. A *reduced word* for a permutation $w \in S_k$ is a word $i_1 i_2 \cdots i_l$ of shortest possible length such that $w = s_{i_1} s_{i_2} \cdots s_{i_l}$. Let $\mathcal{R}(w)$ be the set of reduced words for $w \in S_k$.

The *length* of w is the length $\ell(w)$ of any its reduced words.

This is also equal to the number of pairs $(i,j) \in [k] \times [k]$ with $i < j$ and $w(i) > w(j)$.

We often write the word $w(1)w(2)\cdots w(k)$ to represent $w \in S_k$.

For example, $\ell(321) = 3$ and $\mathcal{R}(321) = \{121, 212\}$ since $321 = (1,2)(2,3)(1,2) = (2,3)(1,2)(2,3)$.

Consider the equivalence relative on words that is the transitive closure of the symmetric relation with

$$\cdots ab \cdots \sim \cdots ba \cdots \qquad \text{and} \qquad \cdots a(a+1)a \cdots \sim \cdots (a+1)a(a+1) \cdots$$

for all integers a, b with $|a - b| > 1$.

The corresponding symbols "$\cdots$" here must mask identical subwords on either side of each relation.

Theorem 3.1 (Matsumoto's theorem)**.** Each set of reduced words $\mathcal{R}(w)$ for $w \in S_k$ is then a single equivalence class under the relation $\sim$. Moreover, a word is a reduced word for some permutation if and only if its $\sim$ equivalence class contains no elements with equal adjacent letters.

This statement can be generalized to arbitrary Coxeter groups.

The permutation $w_0 = k \cdots 321$ is the unique element of maximal length in S_k. It has $\ell(w_0) = \binom{k}{2}$.

For $k = 2$ we have $|\mathcal{R}(w_0)| = 1$, which is the number of standard tableaux of shape

$$\lambda = (1) = \begin{array}{|c|}\hline \\ \hline \end{array}.$$

For $k = 3$ we have $|\mathcal{R}(w_0)| = 2$, which is the number of standard tableaux of shape

$$\lambda = (2,1) = \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \end{array}.$$

For $k = 4$ we have $|\mathcal{R}(w_0)| = 16$, which is the number of standard tableaux of shape

$$\lambda = (3,2,1) = \begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1} \end{array}.$$

We will see shortly that this pattern continues.

When $k = 5$ we have $|\mathcal{R}(w_0)| = 768$ and when $k = 6$ we have $|\mathcal{R}(w_0)| = 292864$.

We introduce another variant of the RSK correspondence, called the *Edelman-Greene correspondence.*

Suppose T is a tableau and a is an integer.

Form a tableau $T \xleftarrow{\mathsf{EG}} a$ as follows (this almost the same as the definition of $T \xleftarrow{\mathsf{RSK}} a$):

- At each stage a number x is inserted into a row, starting with a into the first row of T.
- If x is greater than all entries in the row then it is added to the end.
- If the row already contains x, then the row is unchanged and $x + 1$ is inserted into the next row.
- Otherwise, let y be the first entry with $x < y$; then replace y by x and insert y into the next row.

For example, we have $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \xleftarrow{\mathsf{EG}} 1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \xleftarrow{\mathsf{EG}} 1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}$.

Given a reduced word $i_1 i_2 \cdots i_l \in \mathcal{R}(w)$ for a permutation $w \in S_k$, define

$$P_{\mathsf{EG}}(i_1 i_2 \cdots i_l) = \emptyset \xleftarrow{\mathsf{EG}} i_1 \xleftarrow{\mathsf{EG}} i_2 \xleftarrow{\mathsf{EG}} \cdots \xleftarrow{\mathsf{EG}} i_l.$$

Define $Q_{\mathsf{EG}}(i_1 i_2 \cdots i_l)$ to be the standard tableau with the same shape as $P_{\mathsf{EG}}(i_1 i_2 \cdots i_l)$ that contains j in the box added by the insertion of the letter i_j.

The map $a \mapsto (P_{\mathsf{EG}}(a), Q_{\mathsf{EG}}(a))$ for $a \in \mathcal{R}(w)$ is called the *Edelman-Greene correspondence.*

Example 3.2. The word 34121 is a reduced word for the permutation $w = 42153 \in S_5$. We have

$$\begin{array}{|c|}\hline 3\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 3&4\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1&4\\\hline 3\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array} = P_{\mathsf{EG}}(34121) \quad\text{and}\quad Q_{\mathsf{EG}}(34121) = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}.$$

We also have $34212 \in \mathcal{R}(w)$ and

$$\begin{array}{|c|}\hline 3\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 3&4\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array} = P_{\mathsf{EG}}(34212) \quad\text{and}\quad Q_{\mathsf{EG}}(34121) = \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}.$$

A tableau is *increasing* if its rows and columns are strictly increasing.

Theorem 3.3. Fix a permutation $w \in S_k$. The map $a \mapsto (P_{\mathsf{EG}}(a), Q_{\mathsf{EG}}(a))$ is a bijection from words $a \in \mathcal{R}(w)$ to pairs of tableaux (P, Q) in which P is increasing with $\mathfrak{row}(P) \in \mathcal{R}(w)$ and Q is standard with the same shape as P.

Proof idea. Construct an inverse map, similar to what we did to invert RSK. □

Corollary 3.4. The number of reduced words for $w_0 = k \cdots 321 \in S_k$ is the number of standard tableaux of shape $\lambda = (k-1, \ldots, 3, 2, 1)$.

Proof. Every reduced word for w_0 has length $\binom{k}{2}$ and involves only letters in $\{1, 2, \ldots, k-1\}$.

There is only one increasing tableau T with $\binom{k}{2}$ boxes that has all entries in $\{1, 2, \ldots, k-1\}$.

This tableau is $\begin{array}{|c|}\hline 1\\\hline\end{array}$ for $k = 2$, $\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$ for $k = 3$, $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}$ for $k = 4$, $\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&3&4\\\cline{1-3} 3&4\\\cline{1-2} 4\\\cline{1-1}\end{array}$ for $k = 5$ and so on.

The EG correspondence is therefore a bijection $\mathcal{R}(w_0) \xrightarrow{\sim} \{T\} \times \mathrm{SYT}(\lambda)$ for $\lambda = (k-1, \ldots, 3, 2, 1)$. □

An n-fold *increasing reduced factorization* of $w \in S_k$ is a tuple $(a^1, a^2, \ldots, a^n)$ where each a^i is a strictly increasing (possibly empty) word such that the concatenation $a^1 a^2 \cdots a^n \in \mathcal{R}(w)$.

Definition 3.5. Let $\mathrm{RF}_n(w)$ denote the set of all n-fold increasing reduced factorizations of $w \in S_k$.

Given $a = (a^1, a^2, \ldots, a^n) \in \mathrm{RF}_n(w)$, define $P_{\mathsf{EG}}(a) = P_{\mathsf{EG}}(a^1 a^2 \cdots a^n)$ and let $Q_{\mathsf{EG}}(a)$ be the tableau with the same shape as $P_{\mathsf{EG}}(a)$ with the letter j in every box added by inserting letters in the factor a^j.

For example, if $a = (34, \emptyset, 2, \emptyset, \emptyset, 12) \in \mathrm{RF}_6(42153)$ then $Q_{\mathsf{EG}}(34212) = \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}$ so $Q_{\mathsf{EG}}(a) = \begin{array}{|c|c|}\hline 1&1\\\hline 3&6\\\hline 6\\\cline{1-1}\end{array}$.

Theorem 3.6. Let $w \in S_k$ be a permutation. The map $a \mapsto (P_{\mathsf{EG}}(a), Q_{\mathsf{EG}}(a))$ is a bijection from increasing reduced factorizations $a \in \mathrm{RF}_n(w)$ to pairs of tableaux (P, Q) in which P is increasing with $\mathfrak{row}(P) \in \mathcal{R}(w)$ and Q is semistandard with the same shape as P and with entries in $\{1, 2, \dots, n\}$.

Proof idea. The argument is similar to how we showed that the RSK correspondence for integer matrices is a bijection to pairs of semistandard tableaux. □

It follows that if $w \in S_k$ then the Edelman-Green correspondence is a bijection

$$\mathrm{RF}_n(w) \xrightarrow{\sim} \bigsqcup_{T \text{ of shape } \lambda} \{T\} \times \mathrm{SSYT}_n(\lambda) \qquad (**)$$

for a finite set of increasing tableaux T.

We view each set $\{T\} \times \mathrm{SSYT}_n(\lambda)$ as a $\mathrm{GL}(n)$ crystal isomorphic to $\mathrm{SSYT}_n(\lambda)$.

We then give $\mathrm{RF}_n(w)$ the $\mathrm{GL}(n)$ crystal structure that makes (**) into a crystal isomorphism. I.e.:

Theorem 3.7. Let $w \in S_k$. There is a unique $\mathrm{GL}(n)$ crystal structure on $\mathrm{RF}_n(w)$ for the weight map

$$\mathbf{wt}(a^1, a^2, \dots, a^n) = (\ell(a^1), \ell(a^2), \dots, \ell(a^n))$$

such that for all $i \in [n-1]$ and $a = (a^1, a^2, \dots, a^n) \in \mathrm{RF}_n(w))$ we have

$$\begin{matrix} P_{\mathsf{EG}}(e_i(a)) = P_{\mathsf{EG}}(a) \\ P_{\mathsf{EG}}(f_i(a)) = P_{\mathsf{EG}}(a) \end{matrix} \quad \text{and} \quad \begin{matrix} Q_{\mathsf{EG}}(e_i(a)) = e_i(Q_{\mathsf{EG}}(a)) \\ Q_{\mathsf{EG}}(f_i(a)) = f_i(Q_{\mathsf{EG}}(a)). \end{matrix}$$

Moreover, this is a Stembridge crystal.

Like the $\mathrm{GL}(n) \times \mathrm{GL}(r)$ bicrystal on $\mathrm{Mat}_{r \times n}(\mathbb{N})$, one can describe the operators e_i and f_i for this crystal directly, in terms of a certain pairing on entries of adjacent factors a^i and a^{i+1} in an increasing reduced factorization a, without reference to the Edelman-Greene correspondence. This is explained in detail in Chapter 10 of Bump and Schilling's book but we will skip the details here.

Is there a natural bicrystal that extends the $\mathrm{GL}(n)$ crystal structure on $\mathrm{RF}_n(w)$? This is an open question.

Definition 3.8. The *Stanley symmetric polynomial* of $w \in S_k$ is

$$F_w(x_1, x_2, \dots, x_n) = \sum_{(a^1, a^2, \dots, a^n) \in \mathrm{RF}_n(w)} x^{\mathbf{wt}(a^1, a^2, \dots, a^n)} \in \mathbb{Z}[x_1, x_2, \dots, x_n].$$

where $\mathbf{wt}(a^1, a^2, \dots, a^n) = (\ell(a^1), \ell(a^2), \dots, \ell(a^n)) \in \mathbb{Z}^n$.

The *Stanley symmetric function* of $w \in S_k$ is $F_w = \lim_{n \to \infty} F_w(x_1, x_2, \dots, x_n)$.

Our formula is used as the definition of $F_{w^{-1}}$. Note that it is not obvious that F_w is symmetric.

Corollary 3.9. Both $F_w(x_1, x_2, \dots, x_n)$ and F_w are Schur positive symmetric polynomials/functions, i.e., each is a nonnegative integer linear combination of functions $s_\lambda(x_1, x_2, \dots, x_n)$ or s_λ, as appropriate.

Proof. It suffices to show that $F_w(x_1, x_2, \dots, x_n)$ is a Schur positive symmetric polynomial.

This holds because $F_w(x_1, x_2, \dots, x_n)$ is the character of the Stembridge crystal $\mathrm{RF}_n(w)$. □

Corollary 3.10. The longest permutation $w_0 = k \cdots 321 \in S_k$ has $F_{w_0} = s_{(k-1, \dots, 3, 2, 1)}$.

Proof. Let $\delta_k = (k-1, \dots, 3, 2, 1)$. Our observations when computing $|\mathcal{R}(w_0)|$ show that

$$\mathrm{RF}_n(w_0) = \{T\} \times \mathrm{SSYT}_n(\delta_k) \cong \mathrm{SSYT}_n(\delta_k)$$

as $\mathrm{GL}(n)$ crystals, so $F_{w_0}(x_1, x_2, \dots, x_n) = s_{\delta_k}(x_1, x_2, \dots, x_n)$ for all n. □