1 Review from last time

Cauchy identity. Suppose $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ are sequences of commuting indeterminates that commute with each other and with another indeterminate t. Then

$$\prod_{i,j\geq 1} (1 - x_i y_j t)^{-1} = \sum_{\lambda \text{ partition}} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) t^{|\lambda|}$$

Cauchy correspondence. Let m, $n > 0$ be positive integers.

There is a decomposition of graded $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C})$-representations

$$\bigvee (\mathbb{C}^n \otimes \mathbb{C}^m) \cong \bigoplus_\lambda \pi_\lambda^{\mathrm{GL}(n)} \otimes \pi_\lambda^{\mathrm{GL}(m)}$$

where λ runs through all partitions of with at most $\min\{m,n\}$ nonzero parts.

Reduced factorizations. A *reduced word* for $w \in S_k$ is a word $i_1 i_2 \cdots i_l$ of shortest possible length such that $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ where $s_i = (i, i+1)$. Let $\mathcal{R}(w)$ be the set of such words. An n-fold *increasing reduced factorization* of $w \in S_k$ is a tuple $(a^1, a^2, \dots, a^n)$ where each a^i is a strictly increasing (possibly empty) word such that the concatenation $a^1 a^2 \cdots a^n \in \mathcal{R}(w)$. Let $\mathrm{RF}_n(w)$ denote the set of such tuples.

Edelman-Greene correspondence. We introduced another variant of the RSK correspondence, called the *Edelman-Greene correspondence.* If $w \in S_k$ then the Edelman-Green correspondence is a bijection

$$\mathrm{RF}_n(w) \xrightarrow{\sim} \bigsqcup_{T \text{ of shape } \lambda} \{T\} \times \mathrm{SSYT}_n(\lambda) \tag{*}$$

for a finite set of tableaux T with increasing rows and columns and with $\mathfrak{row}(T)$ a reduced word for w.

Stanley symmetric functions. We gave $\mathrm{RF}_n(w)$ the $\mathrm{GL}(n)$ crystal structure that makes (*) into a crystal isomorphism where $\{T\} \times \mathrm{SSYT}_n(\lambda) \cong \mathrm{SSYT}_n(\lambda)$. We then defined the *Stanley symmetric polynomial* of $w \in S_k$ as $F_w(x_1, x_2, \dots, x_n) = \mathrm{ch}(\mathrm{RF}_n(w))$. This is Schur positive since $\mathrm{RF}_n(w)$ is a Stembridge $\mathrm{GL}(n)$ crystal. The *Stanley symmetric function* of $w \in S_k$ is the symmetric power series

$$F_w = \lim_{n\to\infty} F_w(x_1, x_2, \dots, x_n).$$

The permutation $w_0 = k \cdots 321 \in S_k$ has $F_{w_0} = s_\delta$ and $|\mathcal{R}(w_0)| = |\mathrm{SYT}(\delta)|$ for $\delta = (k-1, \dots, 3, 2, 1)$.

2 String patterns

This lecture corresponds to Chapter 11 in Bump and Schilling's book.

Let (Φ, Λ) be a Cartan type with simple roots $\{\alpha_i : i \in I\}$. Write W for the Weyl group of Φ.

Recall that $W = \langle s_i : i \in I\rangle$ where s_i is the reflection through α_i mapping $x \mapsto x - \langle x, \alpha_i^\vee\rangle \alpha_i$.

The group W is finite and it contains a unique longest element w_0 mapping $\Phi^+ \to \Phi^-$.

In type A_{n-1} we have $W = S_n$ and $w_0 = n \cdots 321$.

Let $\mathbf{i} = (i_1, \dots, i_N)$ be a reduced word for w_0. That is, a sequence of minimal length with $w_0 = s_{i_1} \cdots s_{i_N}$.

Now suppose $\mathcal{C}$ is a seminormal crystal of Cartan type (Φ, Λ). Fix an element $v \in \mathcal{C}$.

We define a sequence of integers $a_j = a_j(v, \mathbf{i}) \geq 0$ as follows.

- First let $a_1 = a_1(v, \mathbf{i})$ be the maximal integer with $e_{i_1}^{a_1}(v) \neq 0$. Note that $a_1 = \varepsilon_{i_1}(v)$.

- Then let a_2 be the maximal integer with $e_{i_2}^{a_2} e_{i_1}^{a_1}(v) \neq 0$.

  $\vdots$

- Then let a_N be the maximal integer with $e_{i_N}^{a_N} \cdots e_{i_2}^{a_2} e_{i_1}^{a_1}(v) \neq 0$.

We call this the *string pattern* of v.

Some ad hoc terminology: $\mathbf{i}$ is a *good word* for $\mathcal{C}$ if $e_{i_N}^{a_N} \cdots e_{i_2}^{a_2} e_{i_1}^{a_1}(v)$ is always a highest weight element.

We will show eventually that if $\mathcal{C}$ is normal then any reduced word for w_0 is a good word.

Our goal today is to prove this fact for $\mathrm{GL}(n)$ crystals.

The idea behind this construction is that the map $v \mapsto (a_1, a_2, \dots, a_N)$ gives us an embedding of $\mathcal{C}$ into a polyhedral cone in $\mathbb{N}^N$. It will turn out that no matter which $\mathbf{i}$ is used, the set of lattice points in the resulting polyhedral cone can be given the structure of a crystal.

The resulting infinite crystal (which does not depend on $\mathbf{i}$) is one realization of the crystal $\mathcal{B}_\infty$ for general Cartan types. We briefly encountered $\mathcal{B}_\infty$ for Cartan type $\mathrm{GL}(2)$ in Lecture 5. The string patterns also indicate a way of embedding each finite normal crystal into $\mathcal{B}_\infty$. This will be explored in Lecture 16.

Assume $\mathbf{i}$ is a good word for the normal $\mathrm{GL}(n)$ crystal of semistandard tableaux $\mathrm{SSYT}_n(\lambda)$.

In this Cartan type, we know from last time that $N = \binom{n}{2}$.

Let $\mathrm{string}_{\mathbf{i}} : \mathrm{SSYT}_n(\lambda) \to \mathbb{N}^N$ be the map

$$\mathrm{string}_{\mathbf{i}}(v) = (a_1(v, \mathbf{i}), a_2(v, \mathbf{i}), \dots, a_N(v, \mathbf{i}))\,.$$

Observation 2.1. The map $\mathrm{string}_{\mathbf{i}}$ is injective: we can recover v from $\mathrm{string}_{\mathbf{i}}(v)$.

Proof. Let $T_\lambda \in \mathrm{SSYT}_n(\lambda)$ be the unique highest weight element (the tableau with all entries i in row i). If $\mathrm{string}_{\mathbf{i}}(v) = (a_1, a_2, \dots, a_N)$ then $T_\lambda = e_{i_N}^{a_N} \cdots e_{i_2}^{a_2} e_{i_1}^{a_1}(v)$, so $v = f_{i_1}^{a_1} f_{i_2}^{a_2} \cdots f_{i_N}^{a_N}(T_\lambda)$. $\square$

We think of the sequence of elements

$$v_0 = v, \quad v_1 = e_{i_1}^{a_1}(v), \quad v_2 = e_{i_2}^{a_2} e_{i_1}^{a_1}(v), \quad \dots, \quad v_N = e_{i_N}^{a_N} \cdots e_{i_2}^{a_2} e_{i_1}^{a_1}(v) = T_\lambda$$

as points on a path through the crystal, which we call the *stations* of v.

Example 2.2. Suppose $n = 3$ so that $W = S_3$ and take $\mathbf{i} = (1, 2, 1)$.

For this word we write string patterns (a_1, a_2, a_3) as upper triangular matrices $\begin{bmatrix} a_2 & a_3 \\ & a_1 \end{bmatrix}$.

Let $\lambda = (4, 2, 0)$ and consider $v = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2} \end{array}$.

Then $e_1^2(v) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 3 & 3 \\ \cline{1-2} \end{array}$ while $e_1^3(v) = 0$ so $a_1(v, \mathbf{i}) = 2$.

We similarly compute that $a_2(v, \mathbf{i}) = 3$ as $e_2^3 e_1^2(v) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$.

Then $a_3(v, \mathbf{i}) = 1$ since $e_1^1 e_2^3 e_1^2(v) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} = T_\lambda$.

Thus we have $\mathrm{string}_{\mathbf{i}}(v) = (2,3,1) = \left[\begin{array}{cc} 3 & 1 \\ & 2 \end{array}\right]$ and the stations of v are

$$v_0 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2}\end{array} \quad\rightsquigarrow\quad v_1 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 3 & 3 \\ \cline{1-2}\end{array} \quad\rightsquigarrow\quad v_2 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \quad\rightsquigarrow\quad v_3 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}.$$

Going from v_0 to v_1 replaces all 2's in the first row by 1's.

Going from v_1 to v_2 replaces all 3's in the first two rows by 2's.

Finally going from v_2 to v_3 replaces all 2's in the first two rows by 1's.

There is a particularly good choice of a good word for $\mathrm{GL}(n)$ crystals. Namely, let

$$\Omega = (1,2,1,3,2,1,4,3,2,1,\dots,n-1,n-2,\dots,3,2,1).$$

This is formed by concatenating the decreasing sequences $(i, i-1, i-2, \dots, 3, 2, 1)$ for $i = 1, 2, \dots, n-1$.

Proposition 2.3. If $\mathbf{i} = (i_1, i_2, \dots, i_N)$ is a reduced word for $w_0 \in S_n$ then so is $(n-i_1, n-i_2, \dots, n-i_N)$.

Proof. If $s_i = (i, i+1)$ then $w_0 s_i w_0^{-1} = (w_0(i), w_0(i+1)) = (n+1-i, n-i) = s_{n-i}$.
Thus if $w_0 = s_{i_1} \cdots s_{i_N}$ then $w_0 = w_0 w_0 w_0^{-1} = w_0 s_{i_1} w_0^{-1} \cdots w_0 s_{i_N} w_0^{-1} = s_{n-i_1} \cdots s_{n-i_N}$. $\square$

Corollary 2.4. The word Ω is a reduced word for $w_0 \in S_n$.

Proof. Recall from last time that there is a unique tableau with strictly increasing rows and columns whose row reading word is a reduced word for $w_0 \in S_n$. When $n = 4$ this tableau is $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$.

If $(i_1, i_2, \dots, i_N)$ is the row reading word for this tableau then $\Omega = (n-i_1, n-i_2, \dots, n-i_N)$.
For example, when $n = 4$ we have $\mathfrak{row}(T) = 323123$ and $\Omega = 121321$. $\square$

Let λ be a partition with at most n nonzero parts.

As in our example, we write string patterns for $\mathbf{i} = \Omega$ as upper triangular arrays

$$\mathrm{string}_\Omega(T) = \left[\begin{array}{cccc} \ddots & & & \vdots \\ & a_4 & a_5 & a_6 \\ & & a_2 & a_3 \\ & & & a_1 \end{array}\right].$$

Fix $T \in \mathrm{SSYT}_n(\lambda)$ and define a_{ij} for $i \geq j$ such that

$$\mathrm{string}_\Omega(T) = \left[\begin{array}{cccc} \ddots & & & \vdots \\ & a_{31} & a_{32} & a_{33} \\ & & a_{21} & a_{22} \\ & & & a_{11} \end{array}\right].$$

Proposition 2.5. The following properties hold:

(i) The number a_{ij} is equal to the number of entries $i+1$ in the top $i+1-j$ rows of T.

(ii) The word Ω is a good word for $\mathrm{SSYT}_n(\lambda)$.

Proof. Renumber the sequence of stations $T_1, T_2, \dots T_N$ as T_{11}, T_{21}, T_{22}, T_{31},... as we did with the a_{ij}.

We will check the following description of T_{ij}. First, T_{11} is obtained from T by replacing all the 2's in the first row by 1's. Then T_{21} is obtained from T_{11} by replacing all 3's in the first two rows of T_{11} by 2's. Then T_{22} is obtained by replacing all 2's in the first row of T_{21} by 1's.

The general pattern is that T_{ij} is obtained from its predecessor by replacing all entries equal to $i+2-j$ in the first $i+1-j$ rows by $i+1-j$.

We explain why this suffices to prove (i) and (ii).

According to our description, the number of replacements in going to T_{ij} from the previous station is the number of entries in that station equal to $i+2-j$. However, the locations of these entries were occupied by $i+1$ in T. Thus this replacement changes a_{ij} entries, where a_{ij} is the number of entries equal to $i+1$ in T. This proves (i).

It is also clear that in the final tableau $T_{n-1,n-1}$, the entries in the ith row are all equal to i, so this tableau is the unique highest weight element. This shows that Ω is a good word as claimed in (ii).

It remains to prove the proposed description of T_{ij}. This is a somewhat technical, but straightforward calculation using the signature rule for crystal operators on tableaux. The full details are in Bump and Schilling's book (see their Proposition 11.2) but we'll skip them in this lecture. $\square$

Proposition 2.6. Let λ be a partition with at most 3 nonzero parts.

Then both reduced words $(1,2,1)$ and $(2,1,2)$ for $w_0 = 321 \in S_3$ are good words for $\mathrm{SSYT}_3(\lambda)$.

Proof. The previous result shows that $\Omega = (1,2,1)$ is a good word. We use a symmetry argument to deduce that the other word $(2,1,2)$ is also good.

The A_2 root system has an automorphism interchanging roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2$ and $\alpha_2 = \mathbf{e}_2 - \mathbf{e}_3$.

Let $\lambda \mapsto \lambda'$ be the corresponding automorphism of the weight lattice.

If $\mathcal{C}$ is an A_2 crystal then we may defined another crystal $\mathcal{C}'$ with the same elements but with weight map $\mathbf{wt}'(v) = \mathbf{wt}(v)'$ and with the indices of the crystal operators e_i and f_i interchanged.

This operation preserves the Stembridge axioms so the new crystal $\mathcal{C}'$ is a Stembridge crystal if and only if $\mathcal{C}$ is Stembridge. Assume this is the case. Then we know that $(1,2,1)$ is a good word for $\mathcal{C}'$, and this implies that $(2,1,2)$ is a good word for $\mathcal{C}$. $\square$

Last time we saw *Matsumoto's theorem* for S_n. We will need the following generalization.

Let $W = \langle s_1, s_2, \dots, s_n \rangle$ be the Weyl group of Φ.

A *reduced word* for $w \in W$ is a minimal length sequence $(i_1, i_2, \dots, i_l)$ with $w = s_{i_1} s_{i_2} \cdots s_{i_l}$.

Define $m(i,j)$ to be the order of the product $s_i s_j \in W$. For Weyl groups this is always finite.

The *reduced word graph* of $w \in W$ is a simple graph whose vertices are the reduced words for w. An edge connects two reduced words in the graph when one word contains a consecutive subsequence $(s_i, s_j, s_i, \dots)$ with $m(i,j)$ terms and the other word is formed by changing this subsequence to $(s_j, s_i, s_j, \dots)$.

Theorem 2.7. The reduced word graph for $w \in W$ is always connected.

If Φ is simply-laced then $m(i,j) = 2$ when α_i and α_j are orthogonal and otherwise $m(i,j) = 3$ if $i \neq j$. In this case we can transform one reduced word for w into any other by a sequence of moves that either swap commuting indices $(i,j) \mapsto (j,i)$ or perform short braid relations $(i,j,i) \mapsto (j,i,j)$.

Proposition 2.8. Suppose $\mathcal{C}$ is a Stembridge crystal for a simply-laced Cartan type. If $\mathcal{C}$ has a good word, then every reduced word for the longest element $w_0 \in W$ is also good.

Proof. By Matsumoto's theorem, it is sufficient to show that if two words $\mathbf{i}$ and $\mathbf{i}'$ are adjacent in the reduced word graph for w_0, and $\mathbf{i}$ is a good word, then $\mathbf{i}'$ is also good. So we can assume that either

$$\mathbf{i} = (i_1, \dots, i_r, i, j, i, i_{r+4}, \dots, i_N) \quad \text{and} \quad \mathbf{i}' = (i_1, \dots, i_r, j, i, j, i_{r+4}, \dots, i_N)$$

when $m(i, j) = 3$ or

$$\mathbf{i} = (i_1, \dots, i_r, i, j, i_{r+4}, \dots, i_N) \quad \text{and} \quad \mathbf{i}' = (i_1, \dots, i_r, j, i, i_{r+4}, \dots, i_N)$$

when $m(i, j) = 2$. We will just give the argument for the first case since the second case is similar.

Let $v \in \mathcal{C}$ and write $v_0 = v$, $v_1 = e_{i_1}^{a_1}(v)$, $v_2 = e_{i_2}^{a_2} e_{i_1}^{a_1}(v)$, $\dots$, $v_N = e_{i_N}^{a_N} \cdots e_{i_2}^{a_2} e_{i_1}^{a_1}(v)$ for the stations of v with respect to $\mathbf{i}$. Similarly let $v = v'_0$, v'_1, v'_2, $\dots$, v'_N be the stations of v with respect to $\mathbf{i}'$.

Assume $\mathbf{i}$ is a good word. Then v_N is a highest weight element.

Since the first r entries of $\mathbf{i}$ and $\mathbf{i}'$ agree, we have $v'_r = v_r$ It suffices to show that $v'_{r+3} = v_{r+3}$.

For this, we branch the crystal to the Levi subsystem A_2 be discarding all edges in the crystal graph except those labeled by i or j. Let $\mathcal{D}$ be the connected component of the branched crystal containing v_r. This is a Stembridge crystal with some highest weight element u.

Our previous proposition shows that both (i, j, i) and (j, i, i) are good words for $\mathcal{D}$. But this means that v_{r+3} and v'_{r+3} must both be equal to u, so $v'_{r+3} = v_{r+3}$ as needed.

Since $\mathbf{i}$ and $\mathbf{i}'$ agree at all letters after $r + 3$, we have $v'_N = v_N$ which is a highest weight element. Using a similar argument for the other case (but branching to $A_1 \times A_1$) we conclude that $\mathbf{i}'$ is a good word. □

Theorem 2.9. Let $\mathcal{C}$ be a connected normal $\mathrm{GL}(n)$ crystal. (Recall that $\mathcal{C}$ is necessarily isomorphic to a twist of $\mathrm{SSYT}_n(\lambda)$ for some λ.) Then every reduced word for $w_0 \in S_n$ is a good word for $\mathcal{C}$.

Proof. This follows since Ω is a good word and the reduced word graph is connected. □

3 Gelfand-Tsetlin patterns

Suppose $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ is a partition with at most n nonzero parts.

Assume $\mu = (\mu_1 \geq \mu_2 \geq \dots \mu_{n-1} \geq 0)$ is a partition with at most $n - 1$ nonzero parts.

The partitions λ and μ *interleave* if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$.

Theorem 3.1. Let λ be a partition with at most n nonzero parts.

Then $\mathrm{SSYT}_n(\lambda)$ has a $\mathrm{GL}(n-1)$ subcrystal isomorphic to $\mathrm{SSYT}_{n-1}(\mu)$ if and only if λ and μ interleave.

When this occurs, $\mathrm{SSYT}_n(\lambda)$ contains a unique a $\mathrm{GL}(n-1)$ subcrystal isomorphic to $\mathrm{SSYT}_{n-1}(\mu)$.

Proof. Assume $T \in \mathrm{SSYT}_n(\lambda)$ is a highest weight element of the branched crystal. This occurs if and only if $e_i(T) = 0$ for all $1 \leq i \leq n - 2$. From the signature rule for the crystal operators, this means that the first row of T can contain only 1's and n's; the second row can contain only 2's and n's, and so forth.

Thus, after eliminating all n's from the tableau T, a tableau T' in the alphabet $[n - 1]$ remains. If μ is the shape of T', then the λ/μ must be a *horizontal strip*, i.e., a skew shape whose diagram never has multiple boxes in the same column, since T cannot have two entries n in the same column. It is easy to see that this is equivalent to assuming that λ and μ interleave.

Since μ is also the $\mathrm{GL}(n-1)$ weight of T, we see that the connected component of the branched crystal containing T is isomorphic to $\mathrm{SSYT}_{n-1}(\mu)$. There is only one such component since it is clear that our original $\mathrm{GL}(n-1)$ highest weight tableau T is uniquely determined by μ and λ. $\square$

A *Gelfand-Tsetlin pattern of size* n is a triangular array

$$\Gamma = \left\{ \begin{array}{ccccccccc} \lambda_{11} & & \lambda_{12} & & \lambda_{13} & & \cdots & & \lambda_{1n} \\ & \lambda_{21} & & \lambda_{22} & & \cdots & & \lambda_{2,n-1} & \\ & & \lambda_{31} & & \cdots & & \lambda_{3,n-2} & & \\ & & & \ddots & & \iddots & & & \\ & & & & \lambda_{n1} & & & & \end{array} \right\}$$

such that each row is a partition and the rows interleave.

Let λ be a partition with at most n parts.

We describe a bijection between Gelfand-Tsetlin patterns Γ whose top row is the partition λ and the crystal $\mathrm{SSYT}_n(\lambda)$. Given $T \in \mathrm{SSYT}_n(\lambda)$, we construct Γ as follows.

The first row of Γ is λ. Removing the boxes labeled $\boxed{n}$ from T gives another tableau T' with entries in $[n-1]$. The shape of T' is the second row of Γ. Then removing the boxes labeled $\boxed{n-1}$ from T' gives another tableau T'' whose shape is the next row of Γ, and so forth.

For example, there are 8 semistandard tableau of shape $\lambda = (2,1,0)$ with entries in $\{1,2,3\}$. The corresponding Gelfand-Tsetlin patterns are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftrightarrow \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array}\right\} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftrightarrow \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{array}\right\}$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftrightarrow \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{array}\right\} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftrightarrow \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}\right\}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftrightarrow \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{array}\right\} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftrightarrow \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array}\right\}$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftrightarrow \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{array}\right\} \qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftrightarrow \left\{\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array}\right\}.$$

In view of the theorem, we may interpret the rows of the Gelfand-Tsetlin pattern associated to T as the height weights for the sequence of branched crystals under the branching from the $\mathrm{GL}(n)$ crystal $\mathrm{SSYT}_n(\lambda)$ to crystals for $\mathrm{GL}(n-1)$, $\mathrm{GL}(n-2)$, etc., that contain T.

The string pattern can also be read off from the Gelfand-Tsetlin pattern. We explain the $n = 3$ case.

Proposition 3.2. Let $n = 3$. If the Gelfand-Tsetlin pattern associated to T is

$$\left\{\begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & a & & b & \\ & & c & & \end{array}\right\}$$

then the string pattern is

$$\mathrm{string}_{(1,2,1)}(T) = \left[\begin{array}{cc} \lambda_1 + \lambda_2 - a - b & \lambda_1 - a \\ & a - c \end{array}\right].$$