1 Last time: string patterns

Let (Φ, Λ) be a Cartan type with simple roots $\{\alpha_i : i \in I\}$. Write W for the Weyl group of Φ.

Let $\mathbf{i} = (i_1, \dots, i_N)$ be a reduced word for the longest element $w_0 \in W$.

Given an element v in a seminormal crystal $\mathcal{C}$ of Cartan type (Φ, Λ):

- First let $a_1 = a_1(v, \mathbf{i})$ be the maximal integer with $e_{i_1}^{a_1}(v) \neq 0$. Note that $a_1 = \varepsilon_{i_1}(v)$.
- Then let a_2 be the maximal integer with $e_{i_2}^{a_2} e_{i_1}^{a_1}(v) \neq 0$.

  $\vdots$
- Then let a_N be the maximal integer with $e_{i_N}^{a_N} \cdots e_{i_2}^{a_2} e_{i_1}^{a_1}(v) \neq 0$.

The sequence $\mathrm{string}_{\mathbf{i}}(v) = (a_1, a_2, \dots, a_N)$ is the *string pattern* of v.

The reduced word $\mathbf{i}$ is a *good word* for $\mathcal{C}$ if $e_{i_N}^{a_N} \cdots e_{i_2}^{a_2} e_{i_1}^{a_1}(v)$ is always a highest weight element.

Our main result last time was the following theorem:

Theorem 1.1. If (Φ, Λ) is Cartan type $\mathrm{GL}(n)$ so that $W = S_n$, and the crystal $\mathcal{C}$ is normal, then every reduced word for $w_0 = n \cdots 321 \in S_n$ is a good word for $\mathcal{C}$.

2 Weyl group actions

Fix a normal crystal $\mathcal{C}$ of Cartan type (Φ, Λ). Continue to assume Φ has simple roots $\{\alpha_i : i \in I\}$.

Our next step is to define an action of the Weyl group $W = \langle s_i : i \in I\rangle$ on $\mathcal{C}$.

Recall that there is an action of W on the weight lattice Λ satisfying $s_i x = x - \langle x, \alpha_i^\vee\rangle \alpha_i$ for $i \in I$.

Back in Lecture 5, we introduced maps $\sigma_i : \mathcal{C} \to \mathcal{C}$ for $i \in I$ by the formula

$$\sigma_i(c) = \begin{cases} f_i^k(c) & \text{if } k \geq 0 \\ e_i^{-k}(c) & \text{if } k < 0. \end{cases} \qquad \text{where } k = \langle \mathbf{wt}(c), \alpha_i^\vee\rangle.$$

We showed that σ_i reverses each i-root string in $\mathcal{C}$ and that $\mathbf{wt}(\sigma_i(c)) = s_i \cdot \mathbf{wt}(c)$ for each $c \in \mathcal{C}$.

Our goal now is to show that the formula $s_i : c \mapsto \sigma_i(c)$ extends to a W-action on $\mathcal{C}$.

This action will automatically satisfy $\mathbf{wt}(w \cdot c) = w \cdot \mathbf{wt}(c)$ for all $w \in W$ and $c \in \mathcal{C}$.

The maps σ_i do not commute the crystal operators, so W won't act as crystal automorphisms, however.

The general form of Matsumuto's theorem says that W is generated by $S = \{s_i : i \in I\}$ subject to the relations $s^2 = 1$ and $sts\cdots = tst\cdots$, both sides have $|st|$ factors, for $s, t \in S$. We know already that each $\sigma_i^2 = 1$, so we just need to show that $\sigma_i\sigma_j\sigma_i \cdots = \sigma_j\sigma_i\sigma_j \cdots$ where both sides have $|s_is_j|$ factors.

Since are working with two operators at a time, we may assume that (Φ, Λ) has rank two, i.e., type $A_1 \times A_1$, A_2, C_2, or G_2. The $A_1 \times A_1$ case is trivial and we will handle the C_2 and G_2 cases by using virtual crystals to leverage the simply-laced case. So the A_2 case will form the major part of our argument.

Proposition 2.1. Let λ be a partition with at most three nonzero parts and suppose $T \in \mathrm{SSYT}_3(\lambda)$.

If the string patterns $\mathrm{string}_{(1,2,1)}(T) = (a_1, a_2, a_3)$ and $\mathrm{string}_{(2,1,2)}(T) = (b_1, b_2, b_3)$ then

$$(b_1, b_2, b_3) = (\max(a_3, a_2 - a_1), a_1 + a_3, \min(a_2 - a_3, a_1)).$$

Proof. From last time, we know that a_1 is the number of 2's in the first row of T, a_2 is the number of 3's in the first two rows, and a_3 is the number of 3's in the first row.

Thus, if R_1, R_2, and R_3 are the three rows of T (considered as words in $\mathbb{B}_n^{\otimes m}$), then

$$a_1 = \varepsilon_1(R_1), \qquad a_2 = \varepsilon_2(R_1) + \varepsilon_2(R_2), \qquad \text{and} \qquad a_3 = \varepsilon_2(R_1).$$

Write $U \in \mathrm{SSYT}_3(\lambda)$ for the unique highest weight element. Then $T = f_1^{a_1} f_2^{a_2} f_1^{a_3}(U) = f_2^{b_1} f_1^{b_2} f_2^{b_3}(U)$.

Therefore $\mathbf{wt}(T) = \lambda - (a_1 + a_3)(\mathbf{e}_1 - \mathbf{e}_2) - a_2(\mathbf{e}_2 - \mathbf{e}_3) = \lambda - (b_1 + b_3)(\mathbf{e}_2 - \mathbf{e}_3) - b_2(\mathbf{e}_1 - \mathbf{e}_2)$.

Comparing the last two expressions gives $b_2 = a_1 + a_3$ and $a_2 = b_1 + b_3$.

Since $b_1 = \varepsilon_2(T) = \varepsilon_2(R_2 \otimes R_1) = \max(\varepsilon_2(R_1), \varepsilon_2(R_1) + \varepsilon_2(R_2) - \varphi_2(R_1))$ and $\varphi_2(R_1) = \varepsilon_1(R_1) = a_1$, we have $b_1 = \max(a_3, a_2 - a_1)$. Finally, $b_3 = a_2 - b_1 = a_2 - \min(a_3, a_2 - a_1) = \max(a_2 - a_3, a_1)$. □

For $N \in \mathbb{Z}$, define maps $\Sigma_N : \mathbb{Z}^3 \to \mathbb{Z}^3$ and $\theta : \mathbb{Z}^3 \to \mathbb{Z}^3$ by

$$\Sigma_N(a, b, c) = (N - a + b - 2c, b, c) \qquad \text{and} \qquad \theta(a, b, c) = (\max(c, b - a), a + c, \min(b - c, a))$$

so that in the previous proposition we have $(b_1, b_2, b_3) = \theta(a_1, a_2, a_3)$.

Proposition 2.2. The composition $\theta \circ \Sigma_M \circ \theta \circ \Sigma_N$ has order 3 for all $M, N \in \mathbb{Z}$.

Proof. One can check that $\Sigma_M^2 = \Sigma_N^2 = \theta^2 = 1$, so we just need to show that $\Sigma_M \circ \theta \circ \Sigma_N \circ \theta \circ \Sigma_M \circ \theta = \theta \circ \Sigma_N \circ \theta \circ \Sigma_M \circ \theta \circ \Sigma_N$. This is a straightforward exercise in algebra. □

We may now prove that the i-string reversing maps σ_i satisfy the nontrivial type A_2 braid relation.

Proposition 2.3. Let λ be a partition with at most three nonzero parts and suppose $T \in \mathrm{SSYT}_3(\lambda)$. Then $\sigma_1\sigma_2\sigma_1(T) = \sigma_2\sigma_1\sigma_2(T)$.

Proof. Let $N = \lambda_1 - \lambda_2$ and $M = \lambda_2 - \lambda_3$. Write $\mathrm{string}(T)$ in place of $\mathrm{string}_{(1,2,1)}(T)$.

The idea is to show that $\mathrm{string}(\sigma_1(T)) = \Sigma_N(\mathrm{string}(T))$ and $\mathrm{string}(\sigma_2(T)) = (\theta \circ \Sigma_M \circ \theta)(\mathrm{string}(T))$.

If this holds then $\mathrm{string} \circ (\sigma_1\sigma_2)^3 = (\theta \circ \Sigma_M \circ \theta \circ \Sigma_N)^3 \circ \mathrm{string} = \mathrm{string}$, which implies that $(\sigma_1\sigma_2)^3 = 1$ since the map $\mathrm{string} : \mathrm{SSYT}_3(\lambda) \to \mathbb{Z}^3$ is injective.

If $\mathrm{string}(T) = (a, b, c)$ then we must have $\mathrm{string}(\sigma_1(T)) = (a', b, c)$ since $\sigma_1(T)$ lies in the 1-root string through T. One can determine a' from the identity $\mathbf{wt}(\sigma_1(T)) = s_1(\mathbf{wt}(T)) = \mathbf{wt}(T) - \langle \mathbf{wt}(T), \alpha_1^\vee \rangle \alpha_1$. There are some details to work out, but this leads to the formula $\mathrm{string}(\sigma_1(T)) = \Sigma_N(\mathrm{string}(T))$.

One can show similarly that $\mathrm{string}_{(2,1,2)}(\sigma_2(T)) = \Sigma_M(\mathrm{string}_{(2,1,2)}(T))$.

The second needed identity then follows by noting that $\mathrm{string}_{(1,2,1)}(T) = \theta \circ \mathrm{string}_{(2,1,2)}(T)$. □

This lets us conclude that if $\mathcal{C}$ is a normal $\mathrm{GL}(3)$ crystal then there is a unique action of S_3 on $\mathcal{C}$ with $s_i \cdot c = \sigma_i(c)$ for $i \in \{1, 2\}$. We extend this to normal crystals of arbitrary Cartan types as follows.

Lemma 2.4.

(C2) Let $\widehat{s}_i = (i, i+1) \in S_4$ for $i \in \{1, 2, 3\}$. Then the elements

$$s_1 := \widehat{s}_1\widehat{s}_3 \qquad s_2 := \widehat{s}_2$$

satisfy the type C_2 Coxeter relations $s_i^2 = 1$ and $s_1s_2s_1s_2 = s_2s_1s_2s_1$.

(G2) Let $\widehat{s}_i = (i, i+1) \in S_5$ for $i \in \{1,2,3,4\}$. Then the elements

$$s_1 := \widehat{s}_1\widehat{s}_3\widehat{s}_4 \qquad s_2 := \widehat{s}_2$$

satisfy the type G_2 Coxeter relations $s_i^2 = 1$ and $s_1s_2s_1s_2s_1s_2 = s_2s_1s_2s_1s_2s_1$.

Proof. This is straightforward: just write down s_1 and s_2 in cycle notation to verify the relations. □

Putting everything together:

Theorem 2.5. Let $\mathcal{C}$ be a normal crystal for a Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$ and Weyl group $W = \langle s_i : i \in I\rangle$. Then there exists a unique action of W on $\mathcal{C}$ in which

$$s_i \cdot c = \sigma_i(c) \quad \text{for each } i \in I \text{ and } c \in \mathcal{C}.$$

Proof. We just need to check that the σ_i operators satisfy the braid relations. We know that $\sigma_i^2 = 1$.

We may assume that (Φ, Λ) has type $A_1 \times A_1$, A_2, C_2, or G_2.

In type $A_1 \times A_1$ a normal crystal must be a disjoint union of rectangles so any two operators σ_i and σ_j commute as needed. In type A_2 the needed relations hold by the previous proposition.

This lets us conclude that in any simply-laced type the σ_i operators satisfy the same braid relations as the simple generators $s_i \in W$.

Assume we are in type C_2. We may assume that $\mathcal{C}$ is a virtual crystal for a type A_3 crystal $\widehat{\mathcal{C}}$. Then $\widehat{\sigma}_1\widehat{\sigma}_3$ acts as σ_1 on $\mathcal{C}$ (since the roots α_1 and α_3 are orthogonal in the A_3 root system) while $\widehat{\sigma}_2$ acts as σ_2. We need to check $((\widehat{\sigma}_1\widehat{\sigma}_3)\widehat{\sigma}_2)^4 = 1$ but this holds since $((\widehat{s}_1\widehat{s}_3)\widehat{s}_2)^4 = 1$ by the lemma.

The G_2 case is similar. We may assume that $\mathcal{C}$ is a virtual crystal for a type D_4 crystal $\widehat{\mathcal{C}}$. Then $\widehat{\sigma}_1\widehat{\sigma}_3\widehat{\sigma}_4$ acts as σ_1 on $\mathcal{C}$ (since the roots α_1, α_3, and α_4 are orthogonal in the D_4 root system) while $\widehat{\sigma}_2$ acts as σ_2, and the braid relation we need to check holds by the lemma. □

3 Motivation for $\mathcal{B}_\infty$

For each Cartan type, there is a crystal $\mathcal{B}_\infty$. The tensor products $\mathcal{T}_\lambda \otimes \mathcal{B}_\infty$ are analogous to the *Verma modules* of the corresponding Lie group of Lie algebra. Verma modules are infinite-dimensional "universal" highest weight modules that can have finite-dimensional irreducible quotients. The crystal $\mathcal{B}_\infty$ is a similar "universal" object that is related to arbitrary normal crystals by crystal morphisms.

Here is a more thorough explanation of this analogy. Let G be a complex analytic Lie group with root system Φ and weight lattice Λ. Let $\mathfrak{g}$ be the Lie algebra of G.

For a weight $\mu \in \Lambda$ let $P(\mu)$ be the number of ways to choose nonnegative integers $k_\alpha \in \mathbb{N}$ for $\alpha \in \Phi^+$ such that $\mu = \sum_{\alpha\in\Phi^+} k_\alpha \alpha$. The function $P : \Lambda \to \mathbb{N}$ is the *Kostant partition function*, and we have

$$\sum_{\mu\in\Lambda} P(\mu)t^{-\mu} = \prod_{\alpha\in\Phi^+} (1 - t^{-\alpha})^{-1}.$$

Let $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}_+$ be the Lie algebra of a Borel subgroup B, where $\mathfrak{t}$ is the Lie algebra of a maximal torus $T \subset B$ and $\mathfrak{n}_+$ is the Lie algebra of the unipotent radical N_+ of B. If $G = \mathrm{GL}(n, \mathbb{C})$ then $\mathfrak{g} = \mathrm{Mat}_{n\times n}(\mathbb{C})$ and we can think of $\mathfrak{b}$, $\mathfrak{t}$, and $\mathfrak{n}_+$ as the sets of upper triangular, diagonal, and strictly upper triangular matrices, respectively.

Identify Λ with the group of regular characters of T, so that by differentiating we can evaluate $\lambda \in \Lambda$ as a linear map $\mathfrak{t} \to \mathbb{C}$. Extend this linear map to a map $\mathfrak{b} \to \mathbb{C}$ that is zero on $\mathfrak{n}_+$ and let $\mathbb{C}_\lambda$ be a 1-dimensional complex vector space on which $\mathfrak{b}$ acts as λ. The *Verma module* of $\lambda \in \Lambda$ is then

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

Here $U(\mathfrak{g})$ denotes the universal enveloping algebra. The $U(\mathfrak{g})$-module $M(\lambda)$ has a quotient that is the irreducible representation of $\mathfrak{g}$ with highest weight λ. One can show that the character of $M(\lambda)$ is

$$\sum_{\mu\in\Lambda} P(\mu)t^{\lambda-\mu} = t^\lambda \prod_{\alpha\in\Phi^+} (1-t^{-\alpha})^{-1}. \qquad (*)$$

The crystal $\mathcal{B}_\infty$ is a "crystal basis" for $M(0)$, and the tensor product $\mathcal{T}_\lambda \otimes \mathcal{B}_\infty$ also has character (*).

Understanding $\mathcal{B}_\infty$ will help us to prove a *refined Demazure character formula* next week.

4 Elementary crystals

We will construct $\mathcal{B}_\infty$ as a subset of tensor products of certain *elementary crystals* $\mathcal{B}_i$.

Fix a Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$. Then fix an index $i \in I$.

Define $\mathcal{B}_i$ as a set to consist of the formal elements $u_i(n)$ for all $n \in \mathbb{Z}$.

The weight of each element is $\mathbf{wt}(u_i(n)) = n\alpha_i$ and the crystal graph is the infinite path

$$\cdots \xrightarrow{i} u_i(2) \xrightarrow{i} u_i(1) \xrightarrow{i} u_i(0) \xrightarrow{i} u_i(-1) \xrightarrow{i} \cdots$$

so $f_i(u_i(n)) = u_i(n-1)$ and e_j and f_j act as zero if $i \neq j$.

The crystal $\mathcal{B}_i$ is not seminormal and its string lengths are

$$\varphi_j(u_i(n)) = \begin{cases} n & \text{if } i = j \\ -\infty & \text{otherwise} \end{cases} \qquad \text{and} \qquad \varepsilon_j(u_i(n)) = \begin{cases} -n & \text{if } i = j \\ -\infty & \text{otherwise.} \end{cases}$$

Recall that if $\lambda \in \Lambda$ then $\mathcal{T}_\lambda = \{t_\lambda\}$ is the 1-element crystal with $\mathbf{wt}(t_\lambda) = \lambda$.

All crystal operators act as zero on $\mathcal{T}_\lambda$ and the string lengths only take value $-\infty$.

If λ is orthogonal to all simple roots that $\mathcal{T}_\lambda \otimes \mathcal{C} \cong \mathcal{C} \otimes \mathcal{T}_\lambda \cong$ ($\mathcal{C}$ twisted by λ).

Without this orthogonality condition, the crystals $\mathcal{T}_\lambda \otimes \mathcal{C}$ and $\mathcal{C} \otimes \mathcal{T}_\lambda$ may not be isomorphic.

This is easy to forget since the maps $x \mapsto t_\lambda \otimes x$ and $x \mapsto x \otimes t_\lambda$ are bijections $\mathcal{C} \to \mathcal{T}_\lambda \otimes \mathcal{C}$ and $\mathcal{C} \to \mathcal{C} \otimes \mathcal{T}_\lambda$ that both shift weights by λ and commute with the relevant crystal operators.

However, the formulas for the string lengths of the tensor products are different: we have

$$\begin{cases} \varphi_j(t_\lambda \otimes x) = \varphi_j(x) + \langle \lambda, \alpha_j^\vee \rangle \\ \varepsilon_j(t_\lambda \otimes x) = \varepsilon_j(x) \end{cases} \qquad \text{compared to} \qquad \begin{cases} \varphi_j(x \otimes t_\lambda) = \varphi_j(x) \\ \varepsilon_j(x \otimes t_\lambda) = \varepsilon_j(x) - \langle \lambda, \alpha_j^\vee \rangle. \end{cases}$$

One always has $\mathcal{T}_\lambda \otimes \mathcal{T}_\mu \cong \mathcal{T}_{\lambda+\mu}$.

The different ways of tensoring the elementary crystal $\mathcal{B}_i$ with $\mathcal{T}_\lambda$ have the following relationship:

Proposition 4.1. If $\lambda \in \Lambda$ then $\mathcal{T}_\lambda \otimes \mathcal{B}_i \cong \mathcal{B}_i \otimes \mathcal{T}_{s_i\lambda}$ where s_i denotes the usual generator of W.

Proof. The desired isomorphism is $t_\lambda \otimes u_i(n) \mapsto u_i(n + \langle \lambda, \alpha_i^\vee \rangle) \otimes t_{s_i\lambda}$. □

Proposition 4.2. It holds that $\mathcal{B}_i \otimes \mathcal{B}_i \cong \bigoplus_{k\in\mathbb{Z}} \mathcal{B}_i \otimes \mathcal{T}_{k\alpha_i} \cong \bigoplus_{k\in\mathbb{Z}} \mathcal{T}_{k\alpha_i} \otimes \mathcal{B}_i$.

Proof. The second two decompositions are isomorphic because $s_i(\alpha_i) = -\alpha_i$.

An isomorphism from $\mathcal{B}_i \otimes \mathcal{B}_i$ to the second direct sum is given by

$$u_i(-n) \otimes u_i(-m) \mapsto t_{-(n+m-k)\alpha_i} \otimes u_i(-k)$$

where $k = \max(n + 2m, m)$. □

5 Constructing $\mathcal{B}_\infty$ for simply-laced types

Complete proofs of the results in this section can be found in §12.2 of Bump and Schilling's book.

We'll skip some of the more technical details in this lecture.

Again write $\theta : \mathbb{Z}^3 \to \mathbb{Z}^3$ for the self-inverse bijection $\theta(a, b, c) = (\max(c, b - a), a + c, \min(b - c, a))$.

Proposition 5.1. Fix indices $i, j \in I$ and suppose α_i and α_j are simple roots.

(i) If α_i and α_j are orthogonal then the map $x \otimes y \mapsto y \otimes x$ is a crystal isomorphism $\mathcal{B}_i \otimes \mathcal{B}_j \xrightarrow{\sim} \mathcal{B}_j \otimes \mathcal{B}_i$. Moreover, in this case the crystal operators e_i, e_j, f_i, f_j do not take the value 0 on $\mathcal{B}_i \otimes \mathcal{B}_j$.

(ii) If $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ then the map

$$\Theta(u_i(-a) \otimes u_j(-b) \otimes u_i(-c)) = u_j(-a') \otimes u_i(-b') \otimes u_j(-c') \quad \text{where } (a', b', c') = \theta(a, b, c)$$

is a crystal isomorphism $\mathcal{B}_i \otimes \mathcal{B}_j \otimes \mathcal{B}_i \xrightarrow{\sim} \mathcal{B}_j \otimes \mathcal{B}_i \otimes \mathcal{B}_j$.

Proof sketch. Part (i) is an easier version of (ii). The map Θ is certainly a bijection and it's easy to check that it is weight preserving. Comparing the general formulas for tensor products with the fairly simple operators for elementary crystals, one checks that Θ preserves the string lengths and commutes with all crystal operators. □

We now specialize to Cartan type A_2, so our index set for the simple roots is $I = \{1, 2\}$.

Let $C = \{(a, b, c) \in \mathbb{Z}^3 : a \geq 0 \text{ and } b \geq c \geq 0\}$. If $(a, b, c) \in C$ then $\theta(a, b, c) \in C$.

Given $a, b, c \in \mathbb{Z}$ define $u(a, b, c) = u_1(-a) \otimes u_2(-b) \otimes u_1(-c) \in \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1$.

Now let $\mathfrak{C} = \{u(a, b, c) : (a, b, c) \in C\}$.

Proposition 5.2. For Cartan type A_2, if $v \in \mathfrak{C}$ then $f_i(v) \in \mathfrak{C}$ and $\varepsilon_i(v) \geq 0$ for all $i \in I$, while $e_i(v) \in \mathfrak{C}$ if and only if $\varepsilon_i(v) > 0$.

Proof. Let $(a', b', c') = \theta(a, b, c)$. Then $\varepsilon_1(u(a, b, c)) = \max(c, a - b + 2c) = b' - c'$ and $\varepsilon_2(u(a, b, c)) = b - c$. Both values are nonnegative if $(a, b, c) \in C = \theta(C)$, so $\varepsilon_i(v) \geq 0$ for all $v \in \mathfrak{C}$.

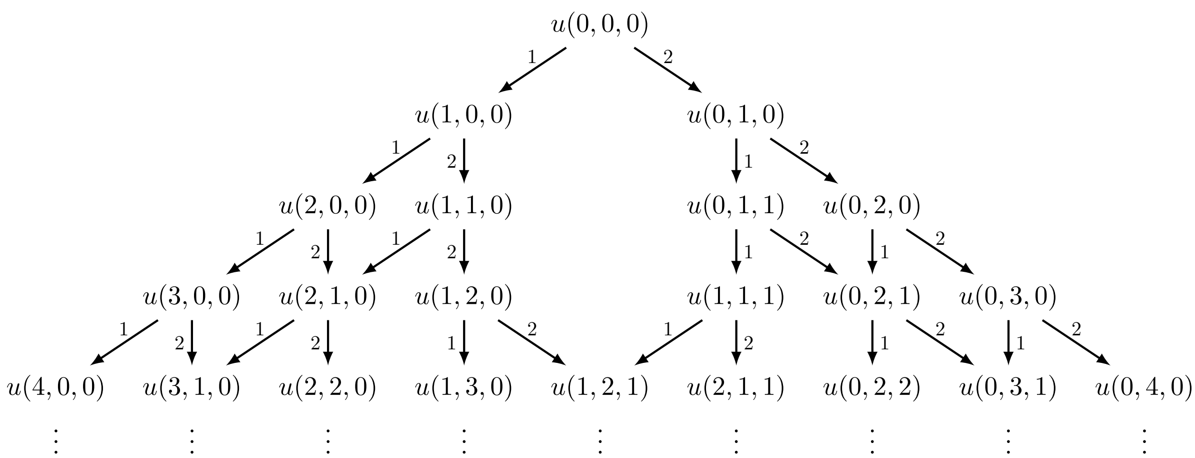
For the remaining assertions, one checks that $e_2(u(a, b, c)) = u(a, b - 1, c)$ and $f_2(u(a, b, c)) = u(a, b + 1, c)$. This implies that $e_2(v) \in \mathfrak{C}$ if and only if $\varepsilon_2(v) > 0$, and that $f_2(v) \in \mathfrak{C}$ for all $v \in \mathfrak{C}$.

We obtain similar statements for e_1 and f_1 by applying θ and using the previous proposition. □

In Cartan type A_2, we define $\mathcal{B}_\infty$ to be the set $\mathfrak{C} \subset \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1$ but with the e_i crystal operators redefined to have $e_i(v) = 0$ if $\varepsilon_i(v) = 0$. The crystal operators f_i, string lengths ε_i and φ_i, and weight map $\mathbf{wt}$ have the same formulas as for $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1$.

With the new definition of e_i, it now holds that $\varepsilon_i(v) = \max\{k \geq 0 : e_i^k(x) \neq 0\}$.

Part of the crystal graph of $\mathcal{B}_\infty$ for Cartan type A_2 is shown below:



We turn next to the construction of $\mathcal{B}_\infty$ for arbitrary simply-laced types.

From this point on we assume our Cartan type (Φ, Λ) is simply-laced, though not necessarily of type A.

Proposition 5.3. Let $w \in W$ be an element of the Weyl group. Suppose

$$w = s_{i_N} \cdots s_{i_2} s_{i_1} = s_{j_N} \cdots s_{j_2} s_{j_1}$$

are two reduced expressions. Then there is a crystal isomorphism

$$\mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N} \xrightarrow{\sim} \mathcal{B}_{j_1} \otimes \cdots \otimes \mathcal{B}_{j_N}$$

mapping $u_{i_1}(0) \otimes \cdots \otimes u_{i_N}(0) \mapsto u_{j_1}(0) \otimes \cdots \otimes u_{j_N}(0)$.

Proof. By Matsumoto's theorem, the desired isomorphism is obtained by composing the isomorphisms in Proposition 5.1, padded appropriately with copies of the identity map on unaffected tensor factors. □

A *weak Stembridge crystal* satisfies the Stembridge axioms but is not required to be seminormal.

Theorem 5.4. Let $w_0 \in W$ be the longest element and suppose $w_0 = s_{i_N} \cdots s_{i_2} s_{i_1}$ is a reduced expression. Then the tensor product $\mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N}$ is a weak Stembridge crystal.

Proof sketch. One must check the Stembridge axioms. The axioms corresponding to orthogonal simple roots are relatively straightforward. For the remaining axioms, a key step is to reduce to the A_2 case by showing that each connected component of $\mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N}$ is isomorphic to $\mathcal{T}_\lambda \otimes \mathcal{B}_i \otimes \mathcal{B}_j \otimes \mathcal{B}_i$ for some λ where $\langle \alpha_i, \alpha_j^\vee \rangle = -1$. Tensoring with $\mathcal{T}_\lambda$ does not affect the Stembridge axioms so one is left to verify that $\mathcal{B}_i \otimes \mathcal{B}_j \otimes \mathcal{B}_i$ is a weak Stembridge crystal. This is a tractable calculation. □

Fix a reduced expression $w_0 = s_{i_N} \cdots s_{i_2} s_{i_1}$ for the longest element in W and define $\mathcal{A} := \mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N}$.

Write $\prec$ for the partial order on $\mathcal{A}$ that is the transitive closure of the relation with $x \prec e_i(x)$ whenever $x \in \mathcal{A}$ and $i \in I$ are such that $e_i(x) \neq 0$. Then let

$$\mathfrak{C} := \{x \in \mathcal{A} : x \preceq u_\infty\} \qquad \text{where } u_\infty := u_{i_1}(0) \otimes \cdots \otimes u_{i_N}(0) \in \mathcal{A}.$$

The set $\mathfrak{C}$ consists of the elements that we can reach by following directed paths starting at u_∞ in the crystal graph for $\mathcal{A}$. This construction is independent of the choice of reduced expression; another choice of reduced expression leads to an isomorphic crystal $\mathcal{A}' \cong \mathcal{A}$ and the isomorphism in Proposition 5.3 maps the analogous subset $\mathfrak{C}' \subset \mathcal{A}'$ to $\mathfrak{C}$.

Proposition 5.5. If $x \in \mathfrak{C}$ then $\varepsilon_i(x) \geq 0$ for all $i \in I$ and if $\varepsilon_i(x) > 0$ then $e_i(x) \in \mathfrak{C}$.

Proof sketch. The proof is similar to the A_2 case, though we must use the formula for the string lengths ε_i of N-fold tensor products, and sometimes appeal to the Stembridge axioms. If we fix $i \in I$, then we may choose the reduced word for w_0 so that $i_N = i$. This simplifies the argument that $\varepsilon_i(x) \geq 0$. □

Here, finally, is the definition of $\mathcal{B}_\infty$ for arbitrary simply-laced types: as a set, we have $\mathcal{B}_\infty = \mathfrak{C}$.

The crystal operators f_i, string lengths ε_i and φ_i, and weight map $\mathbf{wt}$ for $\mathcal{B}_\infty$ are inherited from $\mathcal{A}$.

The value of $e_i(x)$ is the same as for $\mathcal{A}$ if $\varepsilon_i(x) > 0$, but if $\varepsilon_i(x) = 0$ then we set $e_i(x) = 0$.

Theorem 5.6. For these operators, the set $\mathcal{B}_\infty$ is a weak Stembridge crystal that is *upper seminormal* in the sense that $\varepsilon_i(v) = \max\{k \geq 0 : e_i^k(x) \neq 0\}$ for all $v \in \mathcal{B}_\infty$ and $i \in I$.

Proof sketch. The claim that $\mathfrak{C}$ is upper seminormal is immediate from the previous proposition. We have already noted that the crystal $\mathcal{A}$ is weakly Stembridge. This fact plus upper seminormality makes it easy to check that $\mathfrak{C}$ is a crystal satisfying the Stembridge axioms. □

A key property is that we can embed any weak Stembridge crystal in an appropriate twist of $\mathcal{B}_\infty$:

Theorem 5.7. Let $\mathcal{C}$ be a connected weak Stembridge crystal (of a simple-laced Cartan type) whose unique highest weight element u_λ has $\mathbf{wt}(u_\lambda) = \lambda$. Then there exists a unique injective crystal morphism $\psi : \mathcal{C} \to \mathcal{T}_\lambda \otimes \mathcal{B}_\infty$ with $\psi(u_\lambda) = t_\lambda \otimes u_\infty$.

Proof sketch. The argument is similar to how one shows that there exists an isomorphism between connected Stembridge crystals with the same highest weight.

The tensor product $\mathcal{T}_\lambda \otimes \mathcal{B}_\infty$ remains a weak Stembridge crystal.

Let Ω be the set of all subsets S of $\mathcal{C}$ such that $u_\lambda \in S$, if $x \in S$ and $e_i(x) \neq 0$ then $e_i(x) \in S$, and there exists a subset $S' \subset \mathcal{T}_\lambda \otimes \mathcal{B}_\infty$ and a weight-preserving bijection $x \mapsto x'$ from S to S' mapping $u_\lambda \mapsto t_\lambda \otimes u_\infty$ such that if $x \in S$ then $e_i(x) \neq 0$ if and only if $e_i(x') \neq 0$ and $(e_i(x))' = e_i(x')$.

The set Ω is nonempty since $\{u_\lambda\} \in \Omega$. We choose $S \in \Omega$ to be maximal and argue by contradiction that $S = \mathcal{C}$. One derives a contradiction by assuming that S is proper and then showing that we can extend the map $x \mapsto x'$ to a larger set $S \sqcup \{z\} \in \Omega$. □

Corollary 5.8. Let $\lambda \in \Lambda^+$ be any dominant weight and let $\mathcal{B}_\lambda$ be a normal (i.e., Stembridge) crystal for a simply-laced Cartan type with unique highest weight λ. Then there exists a unique injective crystal morphism $\psi_\lambda : \mathcal{B}_\lambda \to \mathcal{T}_\lambda \otimes \mathcal{B}_\infty$ that sends the unique highest weight element to $t_\lambda \otimes u_\infty$

Next time, well discuss $\mathcal{B}_\infty$ for non-simply-laced Cartan types and *Demazure crystals*.