1 Last time: $\mathcal{B}_\infty$ for simply-laced Cartan types

Fix a Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$.

For each index $i \in I$, the *elementary crystal* $\mathcal{B}_i$ is the set of formal elements $u_i(n)$ for all $n \in \mathbb{Z}$.

The weight map of $\mathcal{B}_i$ is $\mathbf{wt}(u_i(n)) = n\alpha_i$ and the crystal graph is the infinite path

$$\cdots \xrightarrow{i} u_i(2) \xrightarrow{i} u_i(1) \xrightarrow{i} u_i(0) \xrightarrow{i} u_i(-1) \xrightarrow{i} \cdots$$

The string lengths of $\mathcal{B}_i$ satisfy $\varphi_i(u_i(n)) = n$ and $\varepsilon_i(u_i(n)) = -n$, with all other values $-\infty$.

Assume (Φ, Λ) is of simply-laced type, meaning that all roots $\alpha \in \Phi$ have the same length.

Fix a reduced expression $w_0 = s_{i_N} \cdots s_{i_2} s_{i_1}$ for the longest element in the Weyl group W.

Define $\mathcal{A} := \mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N}$ and write $\prec$ for the partial order on $\mathcal{A}$ that is the transitive closure of the relation with $x \prec e_i(x)$ whenever $x \in \mathcal{A}$ and $i \in I$ are such that $e_i(x) \neq 0$. Then let

$$\mathfrak{C} := \{x \in \mathcal{A} : x \preceq u_\infty\} \qquad \text{where } u_\infty := u_{i_1}(0) \otimes u_{i_2}(0) \otimes \cdots \otimes u_{i_N}(0) \in \mathcal{A}.$$

Any other choice of reduced expression leads to an isomorphic crystal $\mathcal{A}' \cong \mathcal{A}$ with $\mathfrak{C}' \cong \mathfrak{C}$.

The crystal $\mathcal{B}_\infty$ is given by $\mathfrak{C}$ with all operators inherited from $\mathcal{A}$, but with $e_i(x) = 0$ when $\varepsilon_i(x) = 0$.

Under this definition $\mathcal{B}_\infty$ is a weak Stembridge crystal with $\varepsilon_i(x) = \max\{k \geq 0 : e_i^k(x) \neq 0\}$.

Example 1.1. We discussed $\mathcal{B}_\infty$ for type A_2 last time. For type A_1, the crystal graph of $\mathcal{B}_\infty$ is just

$$u(0) \xrightarrow{1} u(-1) \xrightarrow{1} u(-2) \xrightarrow{1} \cdots$$

writing $u(-n)$ for $u_1(-n)$. Here $\mathbf{wt}(u(-n)) = -n\mathbf{e}_1 + n\mathbf{e}_2$ and $\varphi_1(u(-n)) = -n$ and $\varepsilon_1(u(-n)) = n$.

We first saw this crystal in Example 2.8 in Lecture 5.

2 $\mathcal{B}_\infty$ for non-simply-laced Cartan types

Our first goal today is to extend the construction of $\mathcal{B}_\infty$ to all Cartan types.

This material corresponds to §12.3 in Bump and Schilling's book. The details are a bit technical, so to an even greater extent than last lecture we will mostly skip the proofs and just sketch the main ideas.

The key steps can be branched to rank 2 cases, which leaves just types C_2 and G_2 to consider:

$$\underset{C_2}{s_1 \overset{4}{\text{---}} s_2} \qquad\qquad \underset{G_2}{s_1 \overset{6}{\text{---}} s_2}$$

In type C_2, the longest element of the Weyl group W has two reduced words $w_0 = s_1s_2s_1s_2 = s_2s_1s_2s_1$.

In type G_2, the longest element of the Weyl group W has two reduced words $w_0 = (s_1s_2)^3 = (s_2s_1)^3$.

These braid relations lift to isomorphisms between corresponding tensor products of elementary crystals:

Proposition 2.1. In type C_2 we have $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \cong \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1$.

In type G_2 we have $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \cong \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1$.

Proof idea. For type C_2, one can show that $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2$ and $\mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1$ are respectively isomorphic to weak virtual crystals inside $(\mathcal{B}_1^{A_3} \otimes \mathcal{B}_3^{A_3} \otimes \mathcal{B}_2^{A_3})^{\otimes 2}$ and $(\mathcal{B}_2^{A_3} \otimes \mathcal{B}_1^{A_3} \otimes \mathcal{B}_3^{A_3})^{\otimes 2}$.

Here, a *weak virtual crystal* is defined in the same way as a virtual crystal, except that the ambient crystal is now only required to be a weak Stembridge crystal.

Both $s_1^{A_3} s_3^{A_3} s_2^{A_3} s_1^{A_3} s_3^{A_3} s_2^{A_3}$ and $s_2^{A_3} s_1^{A_3} s_3^{A_3} s_2^{A_3} s_1^{A_3} s_3^{A_3}$ are reduced words for the longest element in S_4.

Therefore by results last time we have $(\mathcal{B}_1^{A_3} \otimes \mathcal{B}_3^{A_3} \otimes \mathcal{B}_2^{A_3})^{\otimes 2} \cong (\mathcal{B}_2^{A_3} \otimes \mathcal{B}_1^{A_3} \otimes \mathcal{B}_3^{A_3})^{\otimes 2}$.

This implies that the weak virtual crystals are also isomorphic. The arguments for type G_2 are similar. □

We can now generalize a result from last time to all Cartan types:

Proposition 2.2. Assume (Φ, Λ) is any Cartan type. Write $W = \langle s_i : i \in I \rangle$ for the Weyl group.

If $w = s_{i_N} \cdots s_{i_2} s_{i_1} = s_{j_N} \cdots s_{j_2} s_{j_1} \in W$ are two reduced words, then there is an isomorphism

$$\mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N} \xrightarrow{\sim} \mathcal{B}_{j_1} \otimes \mathcal{B}_{j_2} \otimes \cdots \otimes \mathcal{B}_{j_N}$$

mapping $u_{i_1}(0) \otimes u_{i_2}(0) \otimes \cdots \otimes u_{i_N}(0) \mapsto u_{j_1}(0) \otimes u_{j_2}(0) \otimes \cdots \otimes u_{j_N}(0)$.

Proof. The argument is the same as in the type A case, using the general form of Matsumoto's theorem along with the isomorphisms for braids relations of length 4 and 6 given in the previous proposition. □

Continue to let (Φ, Λ) be any Cartan type. The definitions of $\mathcal{A}$, $\mathfrak{C} \subset \mathcal{A}$, and $\mathcal{B}_\infty$ are the same as before.

Choose a fixed reduced word $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ for the longest element in W. Let $\mathcal{A} = \mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N}$.

Write $u_\infty := u_{i_1}(0) \otimes u_{i_2}(0) \otimes \cdots \otimes u_{i_N}(0)$ and set $\mathfrak{C} = \{x \in \mathcal{A} : x \preceq u_\infty\}$.

Proposition 2.3. If $x \in \mathfrak{C}$ then $\varepsilon_i(x) \geq 0$ for all $i \in I$, and if $\varepsilon_i(x) > 0$ then $e_i(x) \in \mathfrak{C}$.

Proof idea. The simply-laced case was shown last time. For the non-simply-laced cases, we embed $\mathfrak{C}$ into a virtual crystal for a simply-laced ambient crystal $\widehat{\mathcal{A}}$ corresponding to a tensor product of elementary crystals labeled by a reduced word for the long element. Then we use the axioms $\varepsilon_i(x) = \frac{1}{\gamma_i} \widehat{\varepsilon}_j(x)$ from the definition of virtual crystals to derive what we need from the simply-laced case. □

Now, we define $\mathcal{B}_\infty$ to be the set $\mathfrak{C}$, with the same crystal operators, string lengths, and weight map as $\mathcal{A}$, except we redefine $e_i(x) = 0$ whenever $x \in \mathfrak{C}$ has $\varepsilon_i(x) = 0$.

Say that a crystal $\mathcal{C}$ is *weak normal* if it is either a weak Stembridge crystal or a weak virtual crystal.

Putting everything together gives this result:

Theorem 2.4. The crystal $\mathcal{B}_\infty$ is a weak normal crystal that is upper seminormal in the sense that

$$\varepsilon_i(x) = \max\{k \geq 0 : e_i^k(x) \neq 0\} \qquad \text{for all } x \in \mathcal{B}_\infty \text{ and } i \in I.$$

Moreover, $\mathcal{B}_\infty$ has a unique highest weight element given by u_∞, which has $\mathbf{wt}(u_\infty) = 0$.

There is one other theorem from last time that generalizes.

Recall that if $\lambda \in \Lambda$ then $\mathcal{T}_\lambda = \{t_\lambda\}$ is the 1-element crystal with $\mathbf{wt}(t_\lambda) = \lambda$.

Theorem 2.5. Let (Φ, Λ) be any Cartan type. Suppose $\lambda \in \Lambda^+$ is a dominant weight and $\mathcal{B}_\lambda$ is a connected normal crystal of Cartan type (Φ, Λ) with highest weight λ. Then there exists a unique injective crystal morphism $\psi_\lambda : \mathcal{B}_\lambda \to \mathcal{T}_\lambda \otimes \mathcal{B}_\infty$ that sends the highest weight element u_λ to $t_\lambda \otimes u_\infty$.

Proof. For non-simply laced types, first realize $\mathcal{B}_\lambda$ and $\mathcal{B}_\infty$ as virtual crystals, and then deduce the result from the analogous statement for the underlying ambient crystals, which are simply-laced. □

3 Demazure crystals

The crystal $\mathcal{B}_\infty$ is infinite, so it does not have a well-defined character.

Nevertheless, $\mathcal{B}_\infty$ has certain natural "truncations," which we call *Demazure crystals*.

These are not actually crystals, but they are finite sets with associated characters.

Such *Demazure characters* are closely related to character formulas for representations of Lie groups.

Fix a Cartan type (Φ, Λ) with Weyl group $W = \langle s_i : i \in I\rangle$.

Fix a reduced word $w = s_{i_r} \cdots s_{i_1}$ for some element of W. It is always possible to complete such an expression to a reduced word for the long element $w_0 = s_{i_N} \cdots s_{i_{r+1}} s_{i_r} \cdots s_{i_1}$.

Realize $\mathfrak{C}$ and $\mathcal{A}$ as $\mathfrak{C} = \{x \in \mathcal{A} : x \preceq u_\infty\}$ where $\mathcal{A} = \mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N}$.

Let $\mathcal{A}(w) = \mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_r} \otimes u_{i_{r+1}}(0) \otimes \cdots \otimes u_{i_N}(0)$.

Once the element w is fixed, these constructions are essentially independent of the choice of reduced word, since the isomorphism class of $\mathcal{A}$ is independent of the choice of reduced word for w_0.

Lemma 3.1. Let $x = u_{i_1}(-a_1) \otimes \cdots \otimes u_{i_r}(-a_r) \otimes u_{i_{r+1}}(0) \otimes \cdots \otimes u_{i_N}(0) \in \mathcal{A}(w) \cap \mathfrak{C}$.

If $f_i(x) \notin \mathcal{A}(w)$ then $\varepsilon_i(x) = 0$.

Proof idea. This an exercise from the formulas for f_i and ε_i in N-fold tensor products. The algebra is a little complicated to write down; see Lemma 12.18 in Bump and Schilling's book for the full story. □

Next, we construct a subset $\mathcal{B}_\infty(w)$ of $\mathcal{B}_\infty$ as follows. If $X \subset \mathcal{B}_\infty$ is any subset then let

$$\mathfrak{D}_i X = \{x \in \mathcal{B}_\infty : e_i^k(x) \in X \text{ for some } k \geq 0\}.$$

With $w = s_{i_r} \cdots s_{i_1}$ a reduced word as before, define

$$\mathcal{B}_\infty(w) = \mathfrak{D}_{i_r} \cdots \mathfrak{D}_{i_1}\{u_\infty\}.$$

We call this set a *Demazure crystal* in $\mathcal{B}_\infty$.

We will see shortly that this set does not depend on the choice of reduced word.

Recall that an *i-root string* in a crystal is the set of all elements that may be obtained by applying e_i or f_i to a given element. Every i-root string S in $\mathcal{B}_\infty$ has a unique highest weight element u_S such that $e_i(u_S) = 0$ and for this element we have $S = \mathfrak{D}_i\{u_S\}$.

Theorem 3.2. The following properties hold:

1. Let $w = s_{i_r} \cdots s_{i_1}$ be a reduced word, completed to a reduced word for $w_0 = s_{i_N} \cdots s_{i_1}$.

   Identify $\mathcal{B}_\infty$ with the set $\mathfrak{C}$ in $\mathcal{A} = \mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N}$. Then $\mathcal{B}_\infty(w) = \mathcal{A}(w) \cap \mathfrak{C}$.
2. The set $\mathcal{B}_\infty(w)$ is independent of the choice of reduced word for w.
3. If S is any root string in $\mathcal{B}_\infty$ then $\mathcal{B}_\infty(w) \cap S$ is either $\varnothing$, S or $\{u_S\}$.
4. When $w = w_0$ we have $\mathcal{B}_\infty(w_0) = \mathcal{B}_\infty$.

Proof. We prove the first property by induction on $\ell(w)$. If $w = 1$ then there is nothing to prove so assume $\ell(w) = r > 0$ and write $w = s_{i_r} w'$ where $\ell(w') = r - 1$. By induction

$$\mathcal{B}_\infty(w') = \mathcal{A}(w') \cap \mathfrak{C} = (\mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_{r-1}} \otimes u_{i_r}(0) \otimes \cdots \otimes u_{i_N}(0)) \cap \mathfrak{C}.$$

The set $\mathcal{B}_\infty(w) = \mathfrak{D}_{i_r}\mathcal{B}_\infty(w')$ consists of all elements of $\mathcal{B}_\infty$ that can be obtained by applying f_{i_r} any number of times to elements of $\mathcal{B}_\infty(w')$. The formula for the operator f_i in a general tensor product tells us that applying f_{i_r} to a generic element

$$x = \cdots \otimes u_i(-a_i) \otimes \cdots \in \mathcal{A}(w') \cap \mathfrak{C}$$

will only increment the parameter a_i for $i \leq r$, so it follows that $\mathcal{B}_\infty(w) \subset \mathcal{A}(w) \cap \mathfrak{C}$.

For the reverse inclusion, consider an element $x \in \mathcal{A}(w) \cap \mathfrak{C}$. If $x \notin \mathcal{A}(w') \cap \mathfrak{C} = \mathcal{B}_\infty(w') \subset \mathcal{B}_\infty(w)$ then $\varepsilon_{i_r}(x) > 0$ and it follows by induction on $\varepsilon_{i_r}(x)$ that $e_{i_r}(x) \in \mathcal{B}_\infty(w)$, so $x \in \mathfrak{D}_{i_r}\mathcal{B}_\infty(w') = \mathcal{B}_\infty(w)$.

This proves the first part. The second part follows from our usual observation that $\mathcal{A}(w)$ does not depend on the choice of reduced word for w.

To show the third part, suppose S is an i-root string in $\mathcal{B}_\infty$. To show that $\mathcal{B}_\infty(w) \cap S$ is $\varnothing$, S or $\{u_S\}$, we must check that if $\mathcal{B}_\infty(w) \cap S$ contains any x with $\varepsilon_i(x) > 0$, then $e_i(x)$ and $f_i(x)$ are in $\mathcal{B}_\infty(w)$. This is easy to see for $e_i(x)$, and for $f_i(x)$ the desired claim is the previous lemma.

Finally, for the last part, we have $\mathcal{B}_\infty(w_0) = \mathcal{A}(w_0) \cap \mathfrak{C}$ since $\mathcal{A}(w_0) = \mathcal{A}$. □

There are also versions of Demazure crystals defined as subsets of normal crystals.

Fix a dominant weight $\lambda \in \Lambda^+$ and let $\mathcal{B}_\lambda$ be a connected normal crystal whose unique highest weight element u_λ has weight λ. Let $w \in W$ and suppose $w = s_{i_r} \cdots s_{i_1}$ a reduced word. Again define

$$\mathcal{B}_\lambda(w) = \mathfrak{D}_{i_r} \cdots \mathfrak{D}_{i_1}\{u_\lambda\}$$

where for a set $X \subset \mathcal{B}_\lambda$ we write $\mathfrak{D}_i X = \{x \in \mathcal{B}_\lambda : e_i^k(x) \in X \text{ for some } k \geq 0\}$.

Theorem 3.3. The following properties hold:

1. The set $\mathcal{B}_\lambda(w)$ is independent of the choice of reduced word for w.
2. If S is any root string in $\mathcal{B}_\lambda$ then $\mathcal{B}_\lambda(w) \cap S$ is either $\varnothing$, S or $\{u_S\}$.
3. When $w = w_0$ we have $\mathcal{B}_\lambda(w_0) = \mathcal{B}_\lambda$.

Proof. Consider the injective crystal morphism $\mathcal{B}_\lambda \to \mathcal{T}_\lambda \otimes \mathcal{B}_\infty$ that sends u_λ to $t_\lambda \otimes u_\infty$.

It is easy to show by induction on the length of w that $\mathcal{B}_\lambda(w)$ is just the inverse image of $\mathcal{B}_\infty(w)$ under this map, so all three parts follow from the previous theorem. □

The *Demazure crystals* $\mathcal{B}_\lambda(w)$ are a family of subsets X that are closed under the operators $\mathfrak{D}_i$.

Note: the second property in the theorem says that if $X = \mathcal{B}_\lambda(w)$ and S is any root string, then the intersection $X \cap S$ is either empty, all of S, or just the highest weight element $\{u_S\}$ within the string. Usually, this property is not preserved by the operators $\mathfrak{D}_i$; if it holds for X, then it may fail for $\mathfrak{D}_i X$.

Corollary 3.4. Suppose $\mathcal{C}$ is either the crystal $\mathcal{B}_\infty$ or the normal crystal $\mathcal{B}_\lambda$ for some dominant weight $\lambda \in \Lambda^+$. Then every reduced word for w_0 is a good word for $\mathcal{C}$.

Proof. This is equivalent to our observation that $\mathcal{B}_\infty(w_0) = \mathcal{B}_\infty$ and $\mathcal{B}_\lambda(w_0) = \mathcal{B}_\lambda$. □

4 Demazure characters

Write $\{\alpha_i : i \in I\}$ for the simple roots of our root system Φ. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$.

Let $\mathcal{E}$ be the free abelian group spanned by the symbols t^μ with $\mu \in \Lambda$.

We write $t^\lambda t^\mu = t^{\lambda+\mu}$ if $\lambda, \mu \in \Lambda$ (in which case $\lambda + \mu \in \Lambda$).

If $i \in I$ then $s_i(x) = x - \langle x, \alpha_i^\vee \rangle \alpha_i$ and $s_i(\lambda+\rho) = \lambda + \rho - (\langle \lambda, \alpha_i^\vee \rangle + 1)\alpha_i$.

Given $\lambda \in \Lambda$, then *Demazure operator* on $\mathcal{E}$ is the linear map with the formula

$$D_i(t^\lambda) := \frac{t^{\lambda+\rho} - t^{s_i(\lambda+\rho)}}{1-t^{-\alpha_i}} t^{-\rho} = \begin{cases} t^\lambda(1 + t^{-\alpha_i} + \cdots + t^{-n\alpha_i}) & \text{if } n = \langle \lambda, \alpha_i^\vee \rangle \geq 0 \\ 0 & \text{if } n = \langle \lambda, \alpha_i^\vee \rangle = -1 \\ -t^\lambda(t^{\alpha_i} + \cdots + t^{(-n-1)\alpha_i}) & \text{if } n = \langle \lambda, \alpha_i^\vee \rangle \leq -2. \end{cases}$$

Each simple reflection $s_i \in W$ acts linearly on $\mathcal{E}$ by $s_i \cdot t^\mu = t^{s_i\mu}$.

Lemma 4.1. The Demazure operators satisfy $D_i^2 = D_i$ and $s_i D_i = D_i$.

Proof. One can show that $s_i(\rho) = \rho - \alpha_i$, so we have $D_i f(t) = \frac{f(t) - t^{-\alpha_i} f(s_i t)}{1-t^{-\alpha_i}}$.

Since $s_i\alpha_i = -\alpha_i$, it follows that $s_i D_i f = D_i f$ so $s_i D_i = D_i$ and also $D_i^2 = D_i$. □

Proposition 4.2. The Demazure operators also satisfy the same braid relations as W, meaning that if $m(i,j)$ is the order of $s_i s_j \in W$ then $D_i D_j D_i \cdots = D_j D_i D_j \cdots$ when both sides have $m(i,j)$ factors.

This can be shown directly but the algebra is complicated (and maybe best handled by a computer).

Matsumoto's theorem and the previous result imply that we may define $D_w = D_{i_k} \cdots D_{i_2} D_{i_1}$ where $w = s_{i_k} \cdots s_{i_2} s_{i_1}$ is any reduced word. Let $\lambda \in \Lambda^+$. The *Demazure character* is $\mathrm{ch}_{\lambda,w}(t) = D_w(t^\lambda)$.

Recall that $w_0 \in W$ is the longest element. The following result can be derived from the Weyl character formula, but this proof is beyond the scope of today's lecture.

Theorem 4.3. Assume G is a complex analytic Lie group with root system Φ and weight lattice Λ. The group G has an irreducible representation V_λ with highest weight λ, and if we interpret t^λ as a character on a maximal torus $T \subset G$, then then $\mathrm{ch}_{\lambda,w_0}(t)$ is the character of V_λ.

Our last topic today is the crystal analogue of the Demazure operators D_i.

Let $\mathcal{B}$ be a crystal with an element $b \in \mathcal{B}$. Let $\mathcal{D}_i$ for $i \in I$ be the linear operator $\mathbb{Z}\mathcal{B} \to \mathbb{Z}\mathcal{B}$ on the free abelian group generated by $\mathcal{B}$ with the formula

$$\mathcal{D}_i(b) = \begin{cases} \displaystyle\sum_{0 \leq k \leq \langle \mathbf{wt}(b), \alpha_i^\vee \rangle} f_i^k(b) & \text{if } \langle \mathbf{wt}(b), \alpha_i^\vee \rangle \geq 0 \\ \displaystyle - \sum_{1 \leq k \leq \langle -\mathbf{wt}(b), \alpha_i^\vee \rangle} e_i^k(b) & \text{if } \langle \mathbf{wt}(b), \alpha_i^\vee \rangle < 0. \end{cases}$$

This lifts the formula for $D_i : \mathcal{E} \to \mathcal{E}$ in the following sense.

Extend the map $b \mapsto t^{\mathbf{wt}(b)}$ on $\mathcal{B}$ by linearity to a map $\mathbf{wt}_t : \mathbb{Z}\mathcal{B} \to \mathcal{E}$.

Then $\mathbf{wt}_t \circ \mathcal{D}_i = D_i \circ \mathbf{wt}_t$. Moreover, one can check that $\mathcal{D}_i^2 = \mathcal{D}_i$.

However, unlike the Demazure operators D_i, the operators $\mathcal{D}_i$ do not satisfy the braid relations for W.

If X is a subset of $\mathcal{B}$ then let $\Sigma(X) \in \mathbb{Z}\mathcal{B}$ be the formal sum of its elements.

Again set $\mathfrak{D}_i X = \{x \in \mathcal{B} : e_i^k(x) \in X \text{ for some } k \geq 0\}$.

Lemma 4.4. Let $X \subset \mathcal{B}$ be a subset such that if S is any i-root string in $\mathcal{B}$ then $X \cap S$ is either empty, all of S, or $\{u_S\}$ where u_S is the unique highest weight element in S. Then $\mathcal{D}_i \Sigma(X) = \Sigma(\mathfrak{D}_i X)$.

Proof. We may assume that X is nonempty and contained in a single i-root string S.

Then $X = \{u_S\}$ or $X = S$ and we need to show that $\mathcal{D}_i\Sigma(X) = \Sigma(S)$.

If $X = \{u_S\}$ then we need to show that $\mathcal{D}_i u_S = \Sigma(S)$ but this follows directly from the definition as

$$S = \{u_S, f_i(u_S), \dots, f_i^{\langle \mathbf{wt}(u_S), \alpha_i^\vee \rangle}(u_S)\}.$$

If $X = S$ then we have $\mathcal{D}_i\Sigma(X) = \mathcal{D}_i^2 u_S = \mathcal{D}_i u_S = \Sigma(S)$ as needed. $\square$

The following theorem is the *refined Demazure character formula* promised last time.

Theorem 4.5. Let $\lambda \in \Lambda^+$ be a dominant weight and choose $w \in W$. Then

$$\sum_{v \in \mathcal{B}_\lambda(w)} t^{\mathbf{wt}(v)} = D_{i_r} \cdots D_{i_1} t^\lambda$$

for any reduced word $w = s_{i_r} \cdots s_{i_1}$.

Proof. We may use the lemma repeatedly to see that $\sum(\mathcal{B}_\lambda(w)) = \mathcal{D}_{i_r} \cdots \mathcal{D}_{i_1} u_\lambda$.

Applying the map $\mathbf{wt}_t$ gives the desired formula since $\mathbf{wt}_t \circ \mathcal{D}_i = D_i \circ \mathbf{wt}_t$. $\square$

Corollary 4.6. Assume G is a complex analytic Lie group with root system Φ and weight lattice Λ. The character of the irreducible representation of G with highest weight λ is the same as the character of the connected normal crystal $\mathcal{B}_\lambda$.

Proof. The character of the irreducible representation is $\mathrm{ch}_{\lambda,w_0}(t) = D_{w_0}(t^\lambda) = \sum_{v \in \mathcal{B}_\lambda(w_0)} t^{\mathbf{wt}(v)}$ which is just $\mathrm{ch}(\mathcal{B}_\lambda)$ since $\mathcal{B}_\lambda(w_0) = \mathcal{B}_\lambda$. $\square$