1 Last time: $\mathcal{B}_\infty$ for arbitrary Cartan types

Fix a Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$.

For each index $i \in I$, the *elementary crystal* $\mathcal{B}_i$ is the set of formal elements $u_i(n)$ for all $n \in \mathbb{Z}$.

The weight map of $\mathcal{B}_i$ is $\mathbf{wt}(u_i(n)) = n\alpha_i$ and the crystal graph is the infinite path

$$\cdots \xrightarrow{i} u_i(2) \xrightarrow{i} u_i(1) \xrightarrow{i} u_i(0) \xrightarrow{i} u_i(-1) \xrightarrow{i} \cdots$$

The string lengths of $\mathcal{B}_i$ satisfy $\varphi_i(u_i(n)) = n$ and $\varepsilon_i(u_i(n)) = -n$, with all other values $-\infty$.

Fix a reduced expression $w_0 = s_{i_N} \cdots s_{i_2} s_{i_1}$ for the longest element in the Weyl group W.

Define $\mathcal{A} := \mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N}$ and write $\prec$ for the partial order on $\mathcal{A}$ that is the transitive closure of the relation with $x \prec e_i(x)$ whenever $x \in \mathcal{A}$ and $i \in I$ are such that $e_i(x) \neq 0$. Then let

$$\mathfrak{C} := \{x \in \mathcal{A} : x \preceq u_\infty\} \qquad \text{where } u_\infty := u_{i_1}(0) \otimes u_{i_2}(0) \otimes \cdots \otimes u_{i_N}(0) \in \mathcal{A}.$$

Any other choice of reduced expression leads to an isomorphic crystal $\mathcal{A}' \cong \mathcal{A}$ with $\mathfrak{C}' \cong \mathfrak{C}$.

The crystal $\mathcal{B}_\infty$ is given by $\mathfrak{C}$ with all operators inherited from $\mathcal{A}$, except we set $e_i(x) = 0$ when $\varepsilon_i(x) = 0$.

Theorem 1.1. The crystal $\mathcal{B}_\infty$ is a weak normal crystal with $\varepsilon_i(x) = \max\{k \geq 0 : e_i^k(x) \neq 0\}$.

Moreover, $\mathcal{B}_\infty$ has a unique highest weight element given by u_∞, which has $\mathbf{wt}(u_\infty) = 0$.

Finally, every reduced word for $w_0 \in W$ is a good word for $\mathcal{B}_\infty$.

2 Today: the $\star$-involution of $\mathcal{B}_\infty$

Choose a Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$ and Weyl group $W = \langle s_i : i \in I \rangle$.

Fix a reduced word $\mathbf{i} = (i_1, i_2, \dots, i_N)$ for the longest element $w_0 \in W$.

This choice produces two different embeddings $\mathcal{B}_\infty \hookrightarrow \mathbb{Z}^N$.

First, $\mathcal{B}_\infty$ is isomorphic to a subcrystal of $\mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N}$ and we define

$$\iota_1(x) = (a_1, a_2, \dots, a_N) \in \mathbb{Z}^N \tag{2.1}$$

for $x = u_{i_1}(-a_1) \otimes u_{i_2}(-a_2) \otimes \cdots \otimes u_{i_N}(-a_N) \in \mathcal{B}_\infty$.

Second, we have the embedding

$$\iota_2(x) = \mathrm{string}_{\mathbf{i}}(x) = (b_1, b_2, \dots, b_N). \tag{2.2}$$

Here $b_1 = \varepsilon_{i_1}(x)$, $b_2 = \varepsilon_{i_2}(e_{i_1}^{b_1}(x))$, $b_3 = \varepsilon_{i_3}(e_{i_2}^{b_2} e_{i_1}^{b_1}(x))$, and so forth.

These embeddings are related by a weight-preserving bijection $\star : \mathcal{B}_\infty \to \mathcal{B}_\infty$ which is our main topic of interest today. This map will be written $x \mapsto x^\star$, it will have order two, and it will hold that

$$\iota_1(x^\star) = \iota_2(x) \quad \text{for all } x \in \mathcal{B}_\infty.$$

Although the definitions of ι_1 and ι_2 depend on the choice of reduced word $\mathbf{i}$, the bijection $\star$ will be independent of this choice. Once $\star$ has been given, we can define new crystal operators of $\mathcal{B}_\infty$ by

$$e_i^\star(x) := e_i(x^\star)^\star, \qquad f_i^\star(x) := f_i(x^\star)^\star, \qquad \varepsilon_i^\star(x) := \varepsilon_i(x^\star), \qquad \text{and} \qquad \varphi_i^\star(x) := \varphi_i(x^\star). \tag{2.3}$$

These operators will give the set $\mathcal{B}_\infty$ a second crystal structure.

The new operators $e_i^\star$ and $f_i^\star$ will commute with e_j and f_j when $i \neq j$ (but not generally when $i = j$).

3 Quick derivations in type A_2

As motivation and to start out with a more concrete discussion, we first describe the $\star$ map for type A_2. Define $\tau, \theta : \mathbb{R}^3 \to \mathbb{R}^3$ by the formulas

$$\tau(a,b,c) = (\max(a-b+2c,c), b, \min(a,b-c)) \quad \text{and} \quad \theta(a,b,c) = (\max(c,b-a), a+c, \min(a,b-c)).$$

Lemma 3.1. The maps τ and θ have order two and commute. They also preserve the set

$$C = \{(a,b,c) \in \mathbb{Z}^3 : a \geq 0, b \geq c \geq 0\}.$$

Proof. We have already see that the map θ has order 2 and preserves C.

It can be checked directly that τ also has order 2 and $\theta\tau(a,b,c) = \tau\theta(a,b,c) = (b-c, a+c, c)$.

To see that τ preserves C, let $C_1 = \{(a,b,c) \in C : a+c \geq b\}$ and $C_2 = \{(a,b,c) \in C : a+c < b\}$.

Then check that $\tau(C_1) = C_1$ and $\tau(C_2) = C_2$. □

We can construct $\mathcal{B}_\infty$ in type A_2 as the subcrystal of elements

$$u(a,b,c) := u_1(-a) \otimes u_2(-b) \otimes u_1(-c) \in \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1$$

with $(a,b,c) \in C$. Recall that we redefine $e_i(x) = 0$ if $\varepsilon_i(x) = 0$ for $x \in \mathcal{B}_\infty$.

Proposition 3.2. Relative to this realization of $\mathcal{B}_\infty$, we have $\mathrm{string}_{(1,2,1)}(u(a,b,c)) = \tau(a,b,c)$.

Proof. Let $(a,b,c) \in C$ and $x = u(a,b,c)$. Then $\mathbf{wt}(x) = -(a+c)\alpha_1 - b\alpha_2$ where $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$.

Let $(a',b',c') = \mathrm{string}_{(1,2,1)}(x)$. Then $u_\infty = e_1^{c'} e_2^{b'} e_1^{a'}(x)$ so $\mathbf{wt}(x) = -(a'+c')\alpha_1 - b'\alpha_2$.

Therefore $a'+c' = a+c$ and $b' = b$. By definition $a' = \varepsilon_1(x)$ and by the formulas for string lengths in tensor products we have $a' = \max(a-b+2c, c)$. The desired formula for $c' = a+c-a'$ follows. □

We now define $\star : \mathcal{B}_\infty \to \mathcal{B}_\infty$ by $u(a,b,c)^\star = u(a',b',c')$ where $(a',b',c') = \tau(a,b,c)$.

Corollary 3.3. Continue to assume we are in Cartan type A_2. Then the $\star$ map just given has order 2 and $\iota_1(x^\star) = \iota_2(x)$ for all $x \in \mathcal{B}_\infty$, where ι_1 and ι_2 are the embeddings (2.1) and (2.2).

Proof. The $\star$ map has order 2 since τ has order 2. If $x = u(a,b,c)$ for $(a,b,c) \in C$ and $(a',b',c') = \tau(a,b,c)$, so that $x^\star = u(a',b',c')$, then $\iota_1(x^\star) = (a',b',c')$ and $\iota_2(x) = \mathrm{string}_{(1,2,1)}(x) = (a',b',c')$. □

We could also realize $\mathcal{B}_\infty$ in type A_2 as a subcrystal of $\mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2$.

In this case we would define $x \mapsto x^\star$ by the same formula.

The map $u(a,b,c) \mapsto u(a',b',c')$ for $(a',b',c') = \theta(a,b,c)$ is an isomorphism $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \xrightarrow{\sim} \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2$.

The $\star$ maps commute with this isomorphism since $\tau\theta = \theta\tau$.

Also, we showed in a previous lecture that $\mathrm{string}_{(2,1,2)}(x) = \theta(\mathrm{string}_{(1,2,1)}(x))$.

It follows that we still have $\iota_1(x^\star) = \iota_2(x)$ for all $x \in \mathcal{B}_\infty \subset \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2$.

In this sense, the definition of $\star$ is independent of the choice of reduced word used to define $\mathcal{B}_\infty$ and ι_1.

4 Constructing $\star$ for general types

Now let (Φ, Λ) be an arbitrary Cartan type.

If $\star$ were given, then the formulas (2.3) would give us a second crystal structure on the set $\mathcal{B}_\infty$.

Our approach to defining $\star$ in general is first to construct these operators $e_i^\star$, $f_i^\star$, $\varepsilon_i^\star$, $\varphi_i^\star$.

We will then identify $\star$ as the unique crystal isomorphism that interchanges these two structures on $\mathcal{B}_\infty$.

Proposition 4.1. There exists a unique morphism of crystals $\psi_i : \mathcal{B}_\infty \to \mathcal{B}_i \otimes \mathcal{B}_\infty$ such that

$$\psi_i(u_\infty) = u_i(0) \otimes u_\infty.$$

Choose a reduced word $(i_1, \dots, i_N)$ for $w_0 \in W$ such that $i_1 = i$ and identify $\mathcal{B}_\infty \subset \mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N}$.

If $x = u_{i_1}(-a_1) \otimes \cdots \otimes u_{i_N}(-a_N) \in \mathcal{B}_\infty$ then $\psi_i(x) = u_i(-a_1) \otimes u_{i_1}(0) \otimes \cdots \otimes u_{i_N}(-a_N)$.

Proof. The embedding ψ_i is unique if it exists since u_∞ generates $\mathcal{B}_\infty$ via applications of the f_j crystal operators. We can embed $\mathcal{B}_i \hookrightarrow \mathcal{B}_i \otimes \mathcal{B}_i$ by

$$u_i(-a) \mapsto \begin{cases} u_i(-a) \otimes u_i(0) & \text{if } a \geq 0 \\ u_i(0) \otimes u_i(-a) & \text{otherwise.} \end{cases}$$

On the other hand, $\mathcal{B}_\infty$ is defined as the set of elements

$$x = u_{i_1}(-a_1) \otimes \cdots \otimes u_{i_N}(-a_N) \in \mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N}$$

such that $x \preceq u_\infty = u_{i_1}(0) \otimes \cdots \otimes u_{i_N}(0)$.

Since $i_1 = i$, the map ψ_i is given by tensoring $\mathcal{B}_i \hookrightarrow \mathcal{B}_i \otimes \mathcal{B}_i$ with $N-1$ copies of the identity morphism. $\square$

Lemma 4.2. Suppose that $i_1 = i$ and $x = u_{i_1}(-a_1) \otimes u_{i_2}(-a_2) \otimes \cdots \otimes u_{i_N}(-a_N) \in \mathcal{B}_\infty$.

Then the element $y = u_{i_1}(0) \otimes u_{i_2}(-a_2) \otimes \cdots \otimes u_{i_N}(-a_N)$ is also in $\mathcal{B}_\infty$.

Proof. This holds since $\psi_i(x) = u_i(-a_1) \otimes y \in \mathcal{B}_i \otimes \mathcal{B}_\infty$. $\square$

Define $\mathcal{B}^i$ to be the subset of $\mathcal{B}_\infty$ consisting of the elements $x \in \mathcal{B}_\infty$ with $\psi_i(x) = u_i(0) \otimes x$.

Proposition 4.3. If $x \in \mathcal{B}^i$ and $e_j(x) \neq 0$ for some $j \in I$ then $e_j(x) \in \mathcal{B}^i$.

If $x \in \mathcal{B}^i$ then $\varphi_i(x) \geq 0$, and we have $\varphi_i(x) > 0$ if and only if $f_i(x) \in \mathcal{B}^i$.

Proof. For the first property, note that if $x \in \mathcal{B}^i$ then $\psi_i(x) = u_i(0) \otimes x$, so $\psi_i(e_j(x)) = e_j(u_i(0) \otimes x)$.

This cannot be $e_j(u_i(0)) \otimes x$ which is $u_i(1) \otimes x$ if $j = i$ and 0 otherwise, since

$$\psi_i(e_j(x)) \preceq \psi_i(u_\infty) = u_i(0) \otimes u_\infty.$$

Hence we must have $\psi_i(e_j(x)) = u_i(0) \otimes e_j(x)$ which means that $e_j(x) \in \mathcal{B}^i$.

For the second property, note that if $x \in \mathcal{B}^i$ then $x = u_{i_1}(0) \otimes z$ for $z = u_{i_2}(-a_2) \otimes \cdots \otimes u_{i_N}(-a_N)$.

The usual formulas for string lengths in tensor products give $\varphi_i(x) = \max(0, \varphi_i(z)) \geq 0$.

Suppose $\varphi_i(x) > 0$. Then $\varphi_i(z) > 0$ and $f_i(x) = u_i(0) \otimes f_i(z)$.

Since $f_i(x) \preceq x \preceq u_\infty$, we have $f_i(x) \in \mathcal{B}_\infty$ and clearly $f_i(x) \in \mathcal{B}^i$.

But if $\varphi_i(x) = 0$ then $f_i(x) = u_i(-1) \otimes z \notin \mathcal{B}^i$. $\square$

We can make $\mathcal{B}^i$ into a crystal by redefining $f_i(x) = 0$ if $\varphi_i(x) = 0$ for $x \in \mathcal{B}^i$.

Also, let $\mathcal{B}_i^+ = \{u_i(-a) : a \geq 0\}$. To make this set into a crystal, we redefine $e_i(u_i(0)) = 0$.

Finally, say that a crystal is *i-seminormal*

$$\varphi_i(x) = \max\{k \geq 0 : f_i^k(x) \neq 0\} \qquad \text{and} \qquad \varepsilon_i(x) = \max\{k \geq 0 : e_i^k(x) \neq 0\}$$

for all elements x.

Proposition 4.4. The following properties hold:

(i) The crystal $\mathcal{B}^i$ is upper seminormal and i-seminormal.

(ii) The inclusion map $\mathcal{B}_i^+ \otimes \mathcal{B}^i \hookrightarrow \mathcal{B}_i \otimes \mathcal{B}_\infty$ is a strict crystal morphism.

(iii) The map ψ_i induces an isomorphism between $\mathcal{B}_\infty$ and $\mathcal{B}_i^+ \otimes \mathcal{B}^i$.

Proof sketch. Part (i) follows since $\mathcal{B}_\infty$ is upper seminormal and since $\mathcal{B}^i \sqcup \{0\}$ is stable under e_i.

To show part (ii), one needs to check that the two definitions of $f_i(x)$, viewing $x \in \mathcal{B}^i$ or $x \in \mathcal{B}_\infty$, do not affect the computation of $f_i(u_i(-a) \otimes x)$ in the tensor products $\mathcal{B}_i^+ \otimes \mathcal{B}_i$ and $\mathcal{B}_i \otimes \mathcal{B}_\infty$. This is fairly straightforward.

For part (iii), it is clear that $\psi_i(\mathcal{B}_\infty) \subset \mathcal{B}_i^+ \otimes \mathcal{B}^i$ and it suffices to show that ψ_i is surjective. One can check this by showing that $\mathcal{B}_i^+ \otimes \mathcal{B}^i$ has a unique highest weight element given by $u_i(0) \otimes u_\infty$. □

Although all of the constructions so far have depended on a particular realization of $\mathcal{B}_\infty$ as a subcrystal $\mathcal{B}_\infty \subset \mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N}$ for a reduced word $\mathbf{i} = (i_1, \ldots, i_N)$ for the longest element $w_0 \in W$ (satisfying $i_1 = i$), everything is equivariant with respect to canonical isomorphism between realizations of $\mathcal{B}_\infty$ for different reduced words. So in this sense $\mathcal{B}^i$, $\mathcal{B}_i^+$, and ψ_i are independent of $\mathbf{i}$.

The modified crystal operators $e_i^\star$, $f_i^\star$, $\varepsilon_i^\star$, $\varphi_i^\star$ are now given as follows.

Proposition 4.5. Let $x \in \mathcal{B}_\infty$ and $y \in \mathcal{B}^i$ such that $\psi_i(x) = u_i(-a) \otimes y$.

Define $\varepsilon_i^\star(x) = a$ and $\varphi_i^\star(x) = a + \langle \mathbf{wt}(x), \alpha_i^\vee \rangle$.

Define $e_i^\star(x)$ and $f_i^\star(x)$ by requiring that

$$\psi_i\left(e_i^\star(x)\right) = \begin{cases} u_i(-(a-1)) \otimes y & \text{if } a > 0 \\ 0 & \text{if } a = 0 \end{cases} \qquad \text{and} \qquad \psi_i(f_i^\star(x)) = u_i(-(a+1)) \otimes y.$$

These operators $e_i^\star$, $f_i^\star$, $\varepsilon_i^\star$, $\varphi_i^\star$ give the set $\mathcal{B}_\infty$ an alternative crystal structure with the same weight map.

Proof. The crystal axioms and the fact that $e_i^\star(x)$ and $f_i^\star(x)$ belong to $\mathcal{B}_\infty \sqcup \{0\}$ are easy to check using Proposition 4.3. □

We write $\mathcal{B}_\infty^\star$ to refer to this alternate crystal structure in place of the usual one on $\mathcal{B}_\infty$.

Proposition 4.6. If $i \neq j$ then the operators $e_i^\star$ and $f_i^\star$ commute with e_j and f_j.

We also have $\varepsilon_j(f_i^\star(x)) = \varepsilon_j(x)$ for all $x \in \mathcal{B}_\infty$.

Proof. This is a corollary of the fact that ψ_i induces an isomorphism $\mathcal{B}_\infty \to \mathcal{B}_i^+ \otimes \mathcal{B}^i$.

Since $\varepsilon_j(u_i(-a)) = -\infty$ we have $e_j(u_i(-a) \otimes x) = u_i(-a) \otimes e_j(x)$ for $u_i(-a) \otimes x \in \mathcal{B}_i^+ \otimes \mathcal{B}^i$.

A similar formula holds for f_j. On the other hand, $e_i^\star$ and $f_i^\star$ just decrement or increment a without affecting x, so these operators commute with e_j and f_j.

We have $\varepsilon_j(f_i^\star(x)) = \varepsilon_j(x)$ since $\mathcal{B}_\infty$ is upper seminormal. □

Proposition 4.7. If $x \in \mathcal{B}_\infty^\star$ is a highest weight element then $x = u_\infty$.

Proof. Let $Y = \{x \in \mathcal{B}_\infty : e_i^\star(x) = 0 \text{ for all } i\} = \bigcap_{i\in I} \mathcal{B}^i$ be the set of highest weight elements in $\mathcal{B}_\infty^\star$. We have seen that if $x \in \mathcal{B}^i$ and $y \in \mathcal{B}_\infty$ and $x \preceq y$ then $y \in \mathcal{B}^i$. Thus if $x \in Y$ and $x \preceq y$ then $y \in Y$. Suppose $x \in Y$ and $x \neq u_\infty$. Then we must have $x = f_i(u_\infty)$ for some i so

$$\psi_i(x) = f_i(u_i(0) \otimes u_\infty) = u_i(-1) \otimes u_\infty$$

which means $\varepsilon_i^\star(x) = 1$, which is a contradiction. □

Let $\mathcal{B}^{\star i} = \{x \in \mathcal{B}_\infty : e_i(x) = 0\}$.

Proposition 4.8. The following properties hold:

(i) If $x \in \mathcal{B}^{\star i}$ and $e_j^\star(x) \neq 0$ for any $j \in I$ then $e_j^\star(x) \in \mathcal{B}^{\star i}$.

(ii) If $x \in \mathcal{B}^{\star i}$ then $\varphi_i^\star(x) \geq 0$, with strict inequality if and only if $f_i^\star(x) \in \mathcal{B}^{\star i}$.

Proof idea. This is similar to Proposition 4.3, though the argument is a little more technical. See Proposition 14.11 in Bump and Schilling's book for the full details. □

Now we make $\mathcal{B}^{\star i}$ into a crystal with crystal operators $e_i^\star$ and $f_i^\star$, by redefining $f_i^\star(x) = 0$ if $\varphi_i^\star(x) = 0$. Then $\mathcal{B}^{\star i}$ is a subcrystal of $\mathcal{B}_\infty^\star$ that is upper seminormal and i-seminormal.

Let $\mathcal{B}_i^\star$ be the same as the crystal $\mathcal{B}_i$, but we denote the crystal operations as $\varepsilon_i^\star$, $\varphi_i^\star$, $e_i^\star$, $f_i^\star$ and its elements as $u_i^\star(-a)$ for $a \in \mathbb{Z}$. Define $\mathcal{B}_i^{\star+}$ to be the subcrystal of elements $u_i^\star(-a) \in \mathcal{B}_i^\star$ with $a \geq 0$.

Next, let $\psi_i^\star : \mathcal{B}_\infty^\star \to \mathcal{B}_i^{\star+} \otimes \mathcal{B}^{\star i}$ be the map given as follows.

Let $x \in \mathcal{B}_\infty^\star$ and $a = \varepsilon_i(x)$ and $y = e_i^a(x)$. We have $e_i(y) = 0$ so $y \in \mathcal{B}^{\star i}$, and we define

$$\psi_i^\star(x) = u_i^\star(-a) \otimes y.$$

Proposition 4.9. Let $x \in \mathcal{B}_\infty$ be such that $\psi_i(x) = u_i(-t) \otimes y$ and $\psi_i^\star(x) = u_i^\star(-v) \otimes z$. Then $\varepsilon_i^\star(x) - \varphi_i(y) = \varepsilon_i(x) - \varphi_i^\star(z)$.

Proof idea. The argument for general types is a rather long and technical calculation, involving the Stembridge axioms in simply-laced types and passage to virtual crystals in non-simply laced types. Such complications arise because we have not yet shown that $\psi_i^\star$ is a crystal morphism, so we can only use the fact that ψ_i is a morphism.

The full proof is after Proposition 14.12 in Bump and Schilling's book. □

We finish today's lecture by the stating the following key results. We will talk about the proofs next time, along with properties of the resulting $\star$ involution.

Proposition 4.10. The map $\psi_i^\star : \mathcal{B}_\infty^\star \to \mathcal{B}_i^{\star+} \otimes \mathcal{B}^{\star i}$ is a morphism for the $\star$ crystal structure.

Theorem 4.11. The crystal $\mathcal{B}_\infty^\star$ is isomorphic to $\mathcal{B}_\infty$. Hence, there exists a unique weight-preserving bijection $\vartheta : \mathcal{B}_\infty \to \mathcal{B}_\infty$ such that for every $i \in I$ we have

$$\varepsilon_i^\star \circ \vartheta = \varepsilon_i, \quad \varphi_i^\star \circ \vartheta = \varphi_i, \quad e_i^\star \circ \vartheta = \vartheta \circ e_i, \quad \text{and} \quad f_i^\star \circ \vartheta = \vartheta \circ f_i.$$

Moreover, the map ϑ has order two.

We now *define* the $\star$-involution of $\mathcal{B}_\infty$ by the formula $x \mapsto x^\star := \vartheta(x)$.