1 Last time: the $\star$-involution of $\mathcal{B}_\infty$

Fix a Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$.

The *elementary crystal* $\mathcal{B}_i$ has weight map of $\mathcal{B}_i$ is $\mathbf{wt}(u_i(n)) = n\alpha_i$ and crystal graph

$$\cdots \xrightarrow{i} u_i(2) \xrightarrow{i} u_i(1) \xrightarrow{i} u_i(0) \xrightarrow{i} u_i(-1) \xrightarrow{i} \cdots$$

Fix a reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ for the longest element $w_0 \in W$.

Define $\mathcal{A} := \mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N}$. Write $x \preceq y$ if $e_{j_m} \cdots e_{j_2} e_{j_1}(x) = y$. Then define

$$\mathcal{B}_\infty := \{x \in \mathcal{A} : x \preceq u_\infty\} \quad \text{where } u_\infty := u_{i_1}(0) \otimes u_{i_2}(0) \otimes \cdots \otimes u_{i_N}(0) \in \mathcal{A},$$

with all operators on $\mathcal{B}_\infty$ inherited from $\mathcal{A}$, except $e_i(x) = 0$ when $\varepsilon_i(x) = 0$. We have an embedding

$$\iota_1(x) = (a_1, a_2, \ldots, a_N) \in \mathbb{Z}^N \quad \text{for } x = u_{i_1}(-a_1) \otimes u_{i_2}(-a_2) \otimes \cdots \otimes u_{i_N}(-a_N) \in \mathcal{B}_\infty. \tag{1.1}$$

We have a second embedding, where if $b_1 = \varepsilon_{i_1}(x)$ and $b_2 = \varepsilon_{i_2}(e_{i_1}^{b_1}(x))$ and so forth, then

$$\iota_2(x) = \mathrm{string}_{\mathbf{i}}(x) = (b_1, b_2, \ldots, b_N). \tag{1.2}$$

Our goal is define a weight-preserving involution $\star : \mathcal{B}_\infty \to \mathcal{B}_\infty$ such that $\iota_1(x^\star) = \iota_2(x)$.

We gave a concrete definition of $\star$ in type A_2 last time along with most of the construction in general.

2 Finishing the construction of the $\star$-involution

We continue from where we left off last time.

The way we want to define $\star$ in general is to describe a second crystal structure $\mathcal{B}_\infty^\star$ on the same underlying set as $\mathcal{B}_\infty$, and then identify $\star$ as the unique weight-preserving isomorphism between these structures.

It will not be obvious from this approach that the map $\star$ is an involution, but we will prove this.

2.1 The crystals $\mathcal{B}^i$ and $\mathcal{B}_i^+$

Our strategy involves several other subcrystals of $\mathcal{B}_i$ and $\mathcal{B}_\infty$. We review the definitions.

Throughout, let $x = u_{i_1}(-a_1) \otimes \cdots \otimes u_{i_N}(-a_N) \in \mathcal{B}_\infty$ be a generic element, so $a_1, \ldots, a_N \geq 0$.

We write $\psi_i : \mathcal{B}_\infty \to \mathcal{B}_i \otimes \mathcal{B}_\infty$ for the unique crystal morphism with $\psi_i(u_\infty) = u_i(0) \otimes u_\infty$.

If $i_1 = i$ (which we can assume without loss of generality), then

$$\psi_i(x) = u_i(-a_1) \otimes u_{i_1}(0) \otimes \cdots \otimes u_{i_N}(-a_N) = u_i(-a_1) \otimes y \quad \text{where } y \in \mathcal{B}_\infty.$$

Define $\mathcal{B}^i$ to be the subset of $x \in \mathcal{B}_\infty$ with $\psi_i(x) = u_i(0) \otimes x$, i.e., with $a_1 = 0$ (assuming $i_1 = i$).

We make $\mathcal{B}^i$ into a crystal by redefining $f_i(x) = 0$ if $\varphi_i(x) = 0$ for $x \in \mathcal{B}^i$.

The crystal $\mathcal{B}^i$ is upper seminormal and i-seminormal, meaning that

$$\varphi_i(x) = \max\{k \geq 0 : f_i^k(x) \neq 0\} \qquad \text{and} \qquad \varepsilon_i(x) = \max\{k \geq 0 : e_i^k(x) \neq 0\}$$

and that $e_j(x) \in \mathcal{B}^i \sqcup \{0\}$ and $\varepsilon_j(x) = \max\{k \geq 0 : f_j^k(x) \neq 0\}$ for all $j \in I$ and $x \in \mathcal{B}^i$.

Also, let $\mathcal{B}_i^+ = \{u_i(-a) : a \geq 0\}$. To make this set into a crystal, we redefine $e_i(u_i(0)) = 0$.

Key fact: the map ψ_i is a crystal isomorphism $\mathcal{B}_\infty \xrightarrow{\sim} \mathcal{B}_i^+ \otimes \mathcal{B}^i$.

The tensor product $\mathcal{B}_i^+ \otimes \mathcal{B}^i$ is a subset of $\mathcal{B}_i \otimes \mathcal{B}_\infty$ and turns out to be equal to the image $\psi_i(\mathcal{B}_\infty)$.

2.2 The crystal $\mathcal{B}_\infty^\star$

Now define $\mathcal{B}_\infty^\star$ to be the same set as $\mathcal{B}_\infty$, but viewed as a crystal relative to these modified operators:

- The weight map **wt** is the same as before.
- For $x \in \mathcal{B}_\infty$ with $\psi_i(x) = u_i(-a) \otimes y$ for some $y \in \mathcal{B}^i$, define $\varepsilon_i^\star(x) = a$ and $\varphi_i^\star(x) = a + \langle \mathbf{wt}(x), \alpha_i^\vee \rangle$.
- Next define $e_i^\star(x)$ and $f_i^\star(x)$ by requiring that

$$\psi_i\left(e_i^\star(x)\right) = \begin{cases} u_i(-(a-1)) \otimes y & \text{if } a > 0 \\ 0 & \text{if } a = 0 \end{cases} \qquad \text{and} \qquad \psi_i(f_i^\star(x)) = u_i(-(a+1)) \otimes y.$$

If $i \neq j$ then the operators $e_i^\star$ and $f_i^\star$ commute with e_j and f_j.

If $i \neq j$ then we also have $\varepsilon_j(f_i^\star(x)) = \varepsilon_j(x)$ for all $x \in \mathcal{B}_\infty$.

The highest weight element of $\mathcal{B}_\infty^\star$ is still u_∞.

2.3 The crystals $\mathcal{B}^{\star i}$ and $\mathcal{B}_i^{\star +}$

Let $\mathcal{B}^{\star i} = \{x \in \mathcal{B}_\infty : e_i(x) = 0\}$.

We make $\mathcal{B}^{\star i}$ into a crystal with crystal operators $e_i^\star$ and $f_i^\star$ by redefining $f_i^\star(x) = 0$ if $\varphi_i^\star(x) = 0$.

Then $\mathcal{B}^{\star i}$ is a subcrystal of $\mathcal{B}_\infty^\star$ that is upper seminormal and i-seminormal.

Let $\mathcal{B}_i^\star = \mathcal{B}_i$ but denote the crystal operators as $\varepsilon_i^\star$, $\varphi_i^\star$, $e_i^\star$, $f_i^\star$ and elements as $u_i^\star(-a)$ for $a \in \mathbb{Z}$.

Define $\mathcal{B}_i^{\star +}$ to be the subcrystal of elements $u_i^\star(-a) \in \mathcal{B}_i^\star$ with $a \geq 0$.

Given $x \in \mathcal{B}_\infty^\star$, let $a = \varepsilon_i(x)$ define $y = e_i^a(x) \in \mathcal{B}^{\star i}$.

Next, let $\psi_i^\star : \mathcal{B}_\infty^\star \to \mathcal{B}_i^{\star +} \otimes \mathcal{B}^{\star i}$ be the map with $\psi_i^\star(x) = u_i^\star(-a) \otimes y$.

Last time we sketched a proof of the following technical lemma:

Lemma 2.1. Suppose $x \in \mathcal{B}_\infty$ has $\psi_i(x) = u_i(-t) \otimes y$ and $\psi_i^\star(x) = u_i^\star(-v) \otimes z$. Then

$$\varepsilon_i^\star(x) - \varphi_i(y) = \varepsilon_i(x) - \varphi_i^\star(z).$$

Last time we also stated the following result. Today we explain the proof.

Proposition 2.2. The map $\psi_i^\star : \mathcal{B}_\infty^\star \to \mathcal{B}_i^{\star +} \otimes \mathcal{B}^{\star i}$ is a morphism for the $\star$ crystal structure.

Proof sketch. We need to show that $\varepsilon_j^\star(x) = \varepsilon_j^\star \psi_i^\star(x)$ and $e_j^\star \psi_i^\star(x) = \psi_i^\star e_j^\star(x)$ for all $x \in \mathcal{B}_\infty^\star$ and $j \in I$.

We also need corresponding statements for the $\varphi_j^\star$ and $f_j^\star$, but these are equivalent by the crystal axioms.

Assume $j \neq i$. Then these identities are fairly direct consequences of the fact that e_i and $e_j^\star$ commute.

Since $\varepsilon_j^\star(u_i^\star(-n)) = -\infty$, the operator $e_j^\star$ always acts on $u_i^\star(-n) \otimes y \in \mathcal{B}_i^{\star +} \otimes \mathcal{B}^{\star i}$ on the second factor.

Also, the relevant crystals and are either upper seminormal or i-seminormal.

For example, this means that if $a := \varepsilon_i(x)$ and $y := e_i^a(x)$ then $a = \varepsilon_i e_j^\star(x)$ and $e_j^\star(y) = e_i^a e_j^\star(x)$ so

$$\psi_i^\star(e_j^\star(x)) = u_i^\star(-a) \otimes e_j^\star(y) = e_j^\star(u_i^\star(-a) \otimes y) = e_j^\star(\psi_i^\star(x)).$$

Instead assume that $i = j$. This case is more involved and we only sketch the argument.

Write $\psi_i(x) = u_i(-t) \otimes y$ and $\psi_i^\star(x) = u_i(-v) \otimes z$ where

$$t = \varepsilon_i^\star(x), \qquad v = \varepsilon_i(x), \qquad y = e_i^{\star t}(x), \qquad \text{and} \qquad z = e_i^v(x).$$

There are two subcases. The first case is $t = \varepsilon_i^\star(x) \leq \varphi_i(y)$. By the technical lemma, $v = \varepsilon_i(x) \leq \varphi_i^\star(z)$.

We already know that ψ_i is crystal morphism, so we have

$$v = \varepsilon_i\psi_i(x) = \max\{\varepsilon_i(y), t + \varepsilon_i(y) - \varphi_i(y)\} = \varepsilon_i(y).$$

Because $t \leq \varphi_i(y)$, each time we apply e_i to $\psi_i(x) = u_i(-t) \otimes y$ it applies to the second component, and

$$\psi_i(z) = \psi_i(e_i^v x) = e_i^v \psi_i(x) = u_i(-t) \otimes e_i^v(y).$$

This means that $\varepsilon_i^\star(z) = t = \varepsilon_i^\star(x)$. On the other hand, since $\varepsilon_i(x) \leq \varphi_i^\star(z)$, we have

$$\varepsilon_i^\star \psi_i^\star(x) = \max\{\varepsilon_i^\star(z), \varepsilon_i(x) + \varepsilon_i^\star(z) - \varphi_i^\star(z)\} = \varepsilon_i^\star(z). \qquad (*)$$

Combining these facts gives $\varepsilon_i^\star(\psi_i^\star(x)) = \varepsilon_i^\star(x)$. This proves the first of our identities.

For the second identity, one argues that $\varepsilon_i(e_i^\star(x)) = v$ and that $\psi_i^\star(e_i^\star(x)) = u_i^\star(-v) \otimes e_i^v(e_i^\star(x))$.

Then it suffices to show that $e_i^v(e_i^\star(x)) = e_i^\star(z)$ since $e_i^\star(\psi_i^\star(x)) = e_i^\star(u_i^\star(-v) \otimes z) = u_i^\star(-v) \otimes e_i^\star(z)$.

The justification of these claims follows by calculations similar to those above.

The other subcase is $t = \varepsilon_i^\star(x) > \varphi_i(y)$ in which case the technical lemma implies $v = \varepsilon_i(x) > \varphi_i^\star(z)$.

One argues now that $\varepsilon_i^\star(z) = \varphi_i(y)$, and using (*) that

$$\varepsilon_i^\star \psi_i^\star(x) = \varepsilon_i(x) + \varepsilon_i^\star(z) - \varphi_i^\star(z) = \varepsilon_i^\star(x) + \varepsilon_i^\star(z) - \varphi_i(y) = \varepsilon_i^\star(x),$$

where the second equality holds by the technical lemma.

To show that $e_i^\star \psi_i^\star(x) = \psi_i^\star e_i^\star(x)$, one argues that $\varepsilon_i(e_i^\star(x)) = v - 1$ and $e_i^{v-1}(e_i^\star(x)) = z$ and

$$\psi_i^\star(e_i^\star(x)) = u_i^\star(-(v-1)) \otimes z$$

since then the right hand side is $e_i^\star \psi_i^\star(x)$. □

Theorem 2.3. The crystal $\mathcal{B}_\infty^\star$ is isomorphic to $\mathcal{B}_\infty$. Hence, there exists a unique weight-preserving bijection $\vartheta : \mathcal{B}_\infty \to \mathcal{B}_\infty$ such that for every $i \in I$ we have

$$\varepsilon_i^\star \circ \vartheta = \varepsilon_i, \qquad \varphi_i^\star \circ \vartheta = \varphi_i, \qquad e_i^\star \circ \vartheta = \vartheta \circ e_i, \qquad \text{and} \qquad f_i^\star \circ \vartheta = \vartheta \circ f_i.$$

Proof. Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced word for the longest element in W.

Since we have embeddings $\psi_i : \mathcal{B}_\infty \hookrightarrow \mathcal{B}_i \otimes \mathcal{B}_\infty$ we have an embedding

$$\mathcal{B}_\infty \hookrightarrow \mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N} \otimes \mathcal{B}_\infty.$$

We may construct $\mathcal{B}_\infty$ as the subcrystal of $\mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N}$ generated by $u_\infty := u_{i_1}(0) \otimes \cdots \otimes u_{i_N}(0)$. We can therefore understand the previous map as an embedding $\mathcal{B}_\infty \hookrightarrow \mathcal{B}_\infty \otimes \mathcal{B}_\infty$ given by $x \mapsto x \otimes u_\infty$. In particular, we always have $f_i(x \otimes u_\infty) = f_i(x) \otimes u_\infty$ since $\varphi_i(u_\infty) \leq \varepsilon_i(x)$.

Thus $\mathcal{B}_\infty$ is isomorphic to a subcrystal contained in the set $\mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N} \otimes u_\infty$.

On the other hand we have similar embeddings $\psi_i^\star : \mathcal{B}_\infty^\star \hookrightarrow \mathcal{B}_i^\star \otimes \mathcal{B}_\infty^\star$ for any $i \in I$ and thus also

$$\mathcal{B}_\infty^\star \hookrightarrow \mathcal{B}_{i_1}^\star \otimes \mathcal{B}_{i_2}^\star \otimes \cdots \otimes \mathcal{B}_{i_N}^\star \otimes \mathcal{B}_\infty^\star,$$

and the image of the second embedding is contained in the set $\mathcal{B}^\star_{i_1} \otimes \mathcal{B}^\star_{i_2} \otimes \cdots \otimes \mathcal{B}^\star_{i_N} \otimes u^\star_\infty$.

But $\mathcal{B}_i$ and $\mathcal{B}^\star_i$ are the same crystals, just written in different notation, so the map

$$\theta : u_{i_1}(-a_1) \otimes \cdots \otimes u_{i_N}(-a_N) \otimes u_\infty \mapsto u^\star_{i_1}(-a_1) \otimes \cdots \otimes u^\star_{i_N}(-a_N) \otimes u^\star_\infty$$

is an isomorphism $\mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N} \otimes u_\infty \xrightarrow{\sim} \mathcal{B}^\star_{i_1} \otimes \mathcal{B}^\star_{i_2} \otimes \cdots \otimes \mathcal{B}^\star_{i_N} \otimes u^\star_\infty$.

Let $\vartheta : \mathcal{B}_\infty \to \mathcal{B}^\star_\infty$ be the map that makes the diagram

$$\begin{array}{ccc} \mathcal{B}_\infty & \hookrightarrow & \mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N} \otimes u_\infty \\ \downarrow \vartheta & & \downarrow \theta \\ \mathcal{B}^\star_\infty & \hookrightarrow & \mathcal{B}^\star_{i_1} \otimes \mathcal{B}^\star_{i_2} \otimes \cdots \otimes \mathcal{B}^\star_{i_N} \otimes u^\star_\infty \end{array}$$

commute. This map is the desired isomorphism. It is well-defined because

- the image of $\mathcal{B}_\infty$ in $\mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N} \otimes \mathcal{B}_\infty$ is generated by $u_\infty \otimes u_\infty$,
- the image of $\mathcal{B}^\star_\infty$ in $\mathcal{B}^\star_{i_1} \otimes \mathcal{B}^\star_{i_2} \otimes \cdots \otimes \mathcal{B}^\star_{i_N} \otimes \mathcal{B}^\star_\infty$ is generated by $u^\star_\infty \otimes u^\star_\infty$, and
- we have $\theta(u_\infty \otimes u_\infty) = u^\star_\infty \otimes u^\star_\infty$.

This ensure that the image of $\mathcal{B}_\infty$ under its embedding is mapped by θ to the image of $\mathcal{B}^\star_\infty$. □

Proposition 2.4. The map $\vartheta : \mathcal{B}_\infty \to \mathcal{B}_\infty$ from the previous theorem has order two.

Proof. To simplify our notation we do not distinguish between $\mathcal{B}_i$ and $\mathcal{B}^\star_i$ in this proof, meaning that we consider $u^\star_i(-t) = u_i(-t)$. One can show from/using the proof of Proposition 2.2 that

$$\psi_i e^\star_i = (e_i \otimes 1)\psi_i, \qquad \psi_i f^\star_i = (f_i \otimes 1)\psi_i, \qquad \psi^\star_i e_i = (e_i \otimes 1)\psi^\star_i, \qquad \text{and} \qquad \psi^\star_i f_i = (f_i \otimes 1)\psi^\star_i. \quad (^{**})$$

The map $\psi^\star_i$ is a crystal morphism $\mathcal{B}^\star_\infty \to \mathcal{B}_i \otimes \mathcal{B}^\star_\infty$.

The map $(1 \otimes \vartheta)\psi_i\vartheta^{-1}$ is another morphism $\mathcal{B}^\star_\infty \to \mathcal{B}_i \otimes \mathcal{B}^\star_\infty$.

Since $\mathcal{B}^\star_\infty$ is connected with highest weight element u_∞, there is at most one morphism $\mathcal{B}^\star_\infty \to \mathcal{B}_i \otimes \mathcal{B}^\star_\infty$.

Hence $(1 \otimes \vartheta)\psi_i\vartheta^{-1} = \psi^\star_i$ and $(1 \otimes \vartheta)\psi_i = \psi^\star_i\vartheta$.

Using (**) and the fact that ϑ is a crystal isomorphism, we compute

$$\psi^\star_i \vartheta^2 f_i = \psi^\star_i \vartheta f^\star_i \vartheta = (1 \otimes \vartheta)\psi_i f^\star_i \vartheta = (f_i \otimes \vartheta)\psi_i \vartheta = (f_i \otimes 1)\psi^\star_i \vartheta^2 = \psi^\star_i f_i \vartheta^2.$$

Since $\psi^\star_i$ is injective, it follows that $\vartheta^2 f_i = f_i \vartheta^2$. Similar computations show that ϑ^2 commutes with the e_i operators and is therefore a crystal isomorphism : $\mathcal{B}_\infty \to \mathcal{B}_\infty$.

But the identity map is the only such morphism, so $\vartheta^2 = 1$. □

Finally, here is our definition of the $\star$-involution:

Definition 2.5. The $\star$-involution of $\mathcal{B}_\infty$ is the map $x \mapsto x^\star := \vartheta(x)$.

3 Properties of the $\star$-involution

Let $\mathbf{i} = (i_1, \dots, i_N)$ be a reduced word for $w_0 \in W$. Identify $\mathcal{B}_\infty$ with a subcrystal of $\mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N}$.

The following shows that $\iota_1(x^\star) = \iota_2(x)$ for all $x \in \mathcal{B}_\infty$ for our two embeddings from (1.1) and (1.2).

Theorem 3.1. If $x \in \mathcal{B}_\infty$ has $x^\star = u_{i_1}(-a_1) \otimes \cdots \otimes u_{i_N}(-a_N)$ then $\mathrm{string}_{\mathbf{i}}(x) = (a_1, \ldots, a_N)$.

Proof. Consider the sequence of embeddings

$$\mathcal{B}_\infty^\star \xrightarrow{\psi_{i_1}^\star} \mathcal{B}_{i_1}^\star \otimes \mathcal{B}_\infty^\star \xrightarrow{1\otimes\psi_{i_2}^\star} \mathcal{B}_{i_1}^\star \otimes \mathcal{B}_{i_2}^\star \otimes \mathcal{B}_\infty^\star \xrightarrow{1\otimes 1\otimes\psi_{i_3}^\star} \cdots$$

ending in $\mathcal{B}_{i_1}^\star \otimes \cdots \otimes \mathcal{B}_{i_N}^\star \otimes \{u_\infty^\star\} \subset \mathcal{B}_{i_1}^\star \otimes \cdots \otimes \mathcal{B}_{i_N}^\star \otimes \mathcal{B}_\infty^\star$.

If $x^\star = u_{i_1}(-a_1) \otimes \cdots \otimes u_{i_N}(-a_N)$ then its image under this map is $u_{i_1}(-a_1) \otimes \cdots \otimes u_{i_N}(-a_N) \otimes u_\infty^\star$.

On the other hand, we have $\psi_{i_1}^\star(x) = u_{i_1}^\star(-a_1) \otimes y$, where $y = e_{i_1}^{a_1} x$ and $a_1 = \varepsilon_{i_1}(x)$.

Then applying $1 \otimes \psi_{i_2}^\star$ gives $u_{i_1}^\star(-a_1) \otimes u_{i_2}^\star(-a_2) \otimes z$ where $u_{i_2}^\star(-a_2) \otimes z = \psi_{i_2}^\star(y)$ and so we have $a_2 = \varepsilon_{i_2}(e_{i_1}^{a_1} x)$ and $z = e_{i_2}^{a_2} e_{i_1}^{a_1} x$. Continuing in this way shows that $\mathrm{string}_{\mathbf{i}}(x) = (a_1, \ldots, a_N)$. ☐

Let $w \in W$ with a reduced word $w = s_{i_r} \cdots s_{i_1}$.

Write $\mathcal{B}_\infty(w)$ for the *Demazure crystal* $\mathfrak{D}_{i_r} \cdots \mathfrak{D}_{i_1}\{u_\infty\}$ where $\mathfrak{D}_i X = \{x \in \mathcal{B}_\infty : e_i^k(x) \in X \text{ for a } k \geq 0\}$.

The $\star$-involution interacts with Demazure crystals in a particularly nice way:

Theorem 3.2. We have $\mathcal{B}_\infty(w^{-1})^\star = \mathcal{B}_\infty(w)$.

Proof. We start with a reduced word $(i_1, \ldots, i_r)$ for $w = s_{i_1} \cdots s_{i_r}$ and complete it to a reduced word for $w_0 = s_{i_1} \cdots s_{i_N}$. Note that the order of the reduced word is reversed from the usual convention.

We proved in an earlier lecture that we identify $\mathcal{B}_\infty(w^{-1}) = \mathcal{A}(w^{-1}) \cap \mathfrak{C}$, where

$$\mathfrak{C} = \{x \in \mathcal{B}_{i_1} \otimes \cdots \mathcal{B}_{i_N} : x \preceq u_\infty\} \quad \text{and} \quad \mathcal{A}(w^{-1}) = \mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_r} \otimes u_{i_{r+1}}(0) \otimes \cdots \otimes u_{i_N}(0).$$

Applying $\star$ and using the previous theorem, we see that $\mathcal{B}_\infty(w^{-1})^\star$ is contained in the set of elements in $\mathcal{B}_\infty$ whose string patterns for $\mathbf{i} = (i_1, \ldots, i_N)$ terminate after r steps. Such elements are in $\mathcal{B}_\infty(w)$.

This means that $\mathcal{B}_\infty(w^{-1})^\star \subseteq \mathcal{B}_\infty(w)$. Inverting w gives $\mathcal{B}_\infty(w)^\star \subseteq \mathcal{B}_\infty(w^{-1})$, and since it follows by applying the involution $\star$ that $\mathcal{B}_\infty(w) \subseteq \mathcal{B}_\infty(w^{-1})^\star$. Hence $\mathcal{B}_\infty(w^{-1})^\star = \mathcal{B}_\infty(w)$. ☐

The image of $\psi_i : \mathcal{B}_\infty \hookrightarrow \mathcal{B}_i \otimes \mathcal{B}_\infty$ is $\{u_i(-a) \otimes y \in \mathcal{B}_i \otimes \mathcal{B}_\infty : \varepsilon_i^\star(y) = 0 \text{ and } a \geq 0\}$.

Recall that there is a unique crystal embedding $\psi_\lambda : \mathcal{B}_\lambda \hookrightarrow \mathcal{T}_\lambda \otimes \mathcal{B}_\infty$.

We can also characterize the image of this map.

Lemma 3.3. Let $\Sigma := \{t_\lambda \otimes x : \varepsilon_i^\star(x) \leq \langle \lambda, \alpha_i^\vee \rangle \text{ for all } i\} \subseteq \mathcal{T}_\lambda \otimes \mathcal{B}_\infty$.

If $t_\lambda \otimes x \in \Sigma$ then $\varphi_i(t_\lambda \otimes x) \geq 0$, with strict inequality if and only if $f_i(t_\lambda \otimes x) \in \Sigma$.

Proof. Choose a reduced word $(i_1, i_2, \ldots, i_N)$ for w_0 with $i = i_1$.

When we identify $\mathcal{B}_\infty$ with a subcrystal of $\mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N}$ we may write

$$t_\lambda \otimes x = t_\lambda \otimes u_i(-a) \otimes u_{i_2}(-a_2) \otimes \cdots \otimes u_{i_N}(-a_N)$$

where $a = \varepsilon_i^\star(x)$. The lemma follows by computing $\varphi_i(t_\lambda \otimes x)$ using the formulas for string lengths in N-fold tensor products applied to the RHS. The details are a little technical though not very involved (relative to other results today); see Lemma 14.18 in Bump and Schilling's book for the full argument. ☐

Theorem 3.4. The image of ψ_λ is $\{t_\lambda \otimes x \in \mathcal{T}_\lambda \otimes \mathcal{B}_\infty : \varepsilon_i^\star(x) \leq \langle \lambda, \alpha_i^\vee \rangle \text{ for all } i\}$.

Proof. It follows by the lemma that if we redefine $f_i(t_\lambda \otimes x) = 0$ when $\varphi_i(t_\lambda \otimes x) = 0$ then the set Σ becomes a crystal that is both upper seminormal (since $\mathcal{B}_\infty$ is) and lower seminormal.

Now we argue that the image of ψ_λ is contained in Σ.

The highest weight element u_λ is mapped to $t_\lambda \otimes u_\infty \in \Sigma$.

If $v \in \mathcal{B}_\lambda$ is not a highest weight element then write $v = f_i(y)$ for some $y \in \mathcal{B}_\lambda$.

By induction $\psi_\lambda(y) \in \Sigma$ and $\varphi_i(\psi_\lambda(y)) = \varphi_i(y) > 0$. Since Σ is seminormal, we have $v = f_i(\psi_\lambda(y)) \in \Sigma$.

This shows that image$(\psi_\lambda) \subseteq \Sigma$.

As Σ has a unique highest weight element $t_\lambda \otimes u_\infty$, it is a connected crystal, so image$(\psi_\lambda) = \Sigma$. $\square$