1 Last time: the $\star$-involution of $\mathcal{B}_\infty$

Fix a Cartan type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$.

Elementary crystals: $\mathcal{B}_i = \boxed{\cdots \xrightarrow{i} u_i(2) \xrightarrow{i} u_i(1) \xrightarrow{i} u_i(0) \xrightarrow{i} u_i(-1) \xrightarrow{i} \cdots}$ with $\mathbf{wt}(u_i(n)) = n\alpha_i$.

Fix a reduced word $\mathbf{i} = (i_1, i_2, \dots, i_N)$ for the longest element $w_0 \in W$.

Define $\mathcal{A} := \mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N}$. Write $x \preceq y$ if $e_{j_m} \cdots e_{j_2} e_{j_1}(x) = y$.

Then $\mathcal{B}_\infty := \{x \in \mathcal{A} : x \preceq u_\infty\}$ for $u_\infty := u_{i_1}(0) \otimes u_{i_2}(0) \otimes \cdots \otimes u_{i_N}(0) \in \mathcal{A}$.

All operators on $\mathcal{B}_\infty$ inherited from $\mathcal{A}$, except $e_i(x) = 0$ when $\varepsilon_i(x) = 0$.

We write $\psi_i : \mathcal{B}_\infty \to \mathcal{B}_i \otimes \mathcal{B}_\infty$ for the unique crystal morphism with $\psi_i(u_\infty) = u_i(0) \otimes u_\infty$.

Define $\mathcal{B}^i$ to be the subset of $x \in \mathcal{B}_\infty$ with $\psi_i(x) = u_i(0) \otimes x$.

We make $\mathcal{B}^i$ into upper seminormal and i-seminormal crystal by redefining $f_i(x) = 0$ if $\varphi_i(x) = 0$.

Also, let $\mathcal{B}_i^+ = \{u_i(-a) : a \geq 0\}$. To make this set into a crystal, we redefine $e_i(u_i(0)) = 0$.

Then ψ_i is a crystal isomorphism $\mathcal{B}_\infty \xrightarrow{\sim} \mathcal{B}_i^+ \otimes \mathcal{B}^i$.

Define $\mathcal{B}_\infty^\star$ to be the same set as $\mathcal{B}_\infty$, with same weight map, but with crystal operators that have

$$\varepsilon_i^\star(x) = a \qquad \text{and} \qquad \varphi_i^\star(x) = a + \langle \mathbf{wt}(x), \alpha_i^\vee \rangle$$

for $x, y \in \mathcal{B}_\infty$ with $\psi_i(x) = u_i(-a) \otimes y$, and

$$\psi_i\left(e_i^\star(x)\right) = \begin{cases} u_i(-(a-1)) \otimes y & \text{if } a > 0 \\ 0 & \text{if } a = 0 \end{cases} \qquad \text{and} \qquad \psi_i(f_i^\star(x)) = u_i(-(a+1)) \otimes y.$$

If $i \neq j$ then $e_i^\star$ and $f_i^\star$ commute with e_j and f_j. Highest weight element of $\mathcal{B}_\infty$ and $\mathcal{B}_\infty^\star$ is still u_∞.

Let $\mathcal{B}^{\star i} = \{x \in \mathcal{B}_\infty : e_i(x) = 0\}$, with $f_i^\star(x) = 0$ if $\varphi_i^\star(x) = 0$.

Then $\mathcal{B}^{\star i}$ is a subcrystal of $\mathcal{B}_\infty^\star$ that is upper seminormal and i-seminormal.

Let $\mathcal{B}_i^\star = \mathcal{B}_i$ but denote the crystal operators as $\varepsilon_i^\star$, $\varphi_i^\star$, $e_i^\star$, $f_i^\star$ and elements as $u_i^\star(-a)$ for $a \in \mathbb{Z}$.

Define $\mathcal{B}_i^{\star +}$ to be the subcrystal of elements $u_i^\star(-a) \in \mathcal{B}_i^\star$ with $a \geq 0$.

Given $x \in \mathcal{B}_\infty^\star$, let $a = \varepsilon_i(x)$ define $y = e_i^a(x) \in \mathcal{B}^{\star i}$.

Next, let $\psi_i^\star : \mathcal{B}_\infty^\star \to \mathcal{B}_i^{\star +} \otimes \mathcal{B}^{\star i}$ be the map with $\psi_i^\star(x) = u_i^\star(-a) \otimes y$.

Then the map $\psi_i^\star : \mathcal{B}_\infty^\star \to \mathcal{B}_i^{\star +} \otimes \mathcal{B}^{\star i}$ is a morphism for the $\star$ crystal structure.

Key fact: The crystal $\mathcal{B}_\infty^\star$ is isomorphic to $\mathcal{B}_\infty$, since iterating ψ_i and $\psi_i^\star$ gives embeddings

$$\mathcal{B}_\infty \hookrightarrow \mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_N} \otimes \mathcal{B}_\infty \qquad \text{and} \qquad \mathcal{B}_\infty^\star \hookrightarrow \mathcal{B}_{i_1}^\star \otimes \mathcal{B}_{i_2}^\star \otimes \cdots \otimes \mathcal{B}_{i_N}^\star \otimes \mathcal{B}_\infty^\star,$$

but $\mathcal{B}_i$ and $\mathcal{B}_i^\star$ are the same crystals, and the images of the embeddings can be identified.

Key fact: Write $x \mapsto x^\star$ for the unique crystal isomorphism $\mathcal{B}_\infty \to \mathcal{B}_\infty^\star$. This map has order two.

2 Commutors

We include some final remarks about the connection between the $\star$-involution and the *commutor* map.

Suppose $\mathcal{C}$ is a connected normal crystal of type (Φ, Λ) with simple roots $\{\alpha_i : i \in I\}$.

Let $\tau : I \to I$ be the permutation such that $w_0(\alpha_i) = -\alpha_{\tau(i)}$.

In type A_{n-1} we have $w_0 = n \cdots 321$ and $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ so $w_0(\alpha_i) = \mathbf{e}_{n+1-i} - \mathbf{e}_{n-i} = -\alpha_{n-i}$.

HW3 contained an exercise about *crystal involutions*.

A crystal involution is a map $S : \mathcal{C} \to \mathcal{C}$ with $\mathbf{wt}(S(x)) = w_0(\mathbf{wt}(x))$ as well as

$$e_i(S(x)) = f_{\tau(i)}(x), \quad f_i(S(x)) = e_{\tau(i)}(x), \quad \varepsilon_i(S(x)) = \varphi_{\tau(i)}(x), \quad \varphi_i(S(x)) = \varepsilon_{\tau(i)}(x)$$

for all $i \in I$ and $x \in \mathcal{B}$. This is sometimes called a *Schützenberger involution* or *Lusztig involution*.

The content of the exercise was to show:

Theorem 2.1. A connected normal crystal $\mathcal{C}$ has a unique Lusztig involution.

Write $S : \mathcal{C} \to \mathcal{C}$ for the Lusztig involution. This map sends the highest weight vector to the lowest weight vector sand interchanges f_i with $e_{\tau(i)}$. When $\mathcal{C}$ is a disconnected normal crystal, we define $S : \mathcal{C} \to \mathcal{C}$ to be the map that restrict to the Lusztig involution on each full subcrystal.

Given dominant weights $\lambda, \mu \in \Lambda^+$, the *commutor* is the map

$$C : \mathcal{B}_\mu \otimes \mathcal{B}_\lambda \to \mathcal{B}_\lambda \otimes \mathcal{B}_\mu$$

with the formula $b \otimes c \mapsto S(S(c) \otimes S(b))$. Here $\mathcal{B}_\mu$ is the connected normal crystal with highest weight μ.

When we want to emphasize the crystals involved, we write $C_{\mathcal{A},\mathcal{B}} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$.

Proposition 2.2. The commutor is an involution and satisfies

$$C_{\mathcal{B}\otimes\mathcal{A},\mathcal{C}} \circ (C_{\mathcal{A},\mathcal{B}} \otimes 1) = C_{\mathcal{A},\mathcal{C}\otimes\mathcal{B}} \circ (1 \otimes C_{\mathcal{B},\mathcal{C}}) : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{B} \otimes \mathcal{A}. \tag{2.1}$$

Proof. Applying either side of the identity to $a \otimes b \otimes c \in \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$ gives $S(S(c) \otimes S(b) \otimes S(a))$. □

The identity in (2.1) is known as the *cactus relation*.

It implies that the category of crystals is a *coboundary category*.

The cactus relation imposes the action of the *cactus group* on the tensor product of finite normal crystals.

The cactus group also acts on the *left cells* in a finite Weyl group, if you know about cells.

The commutor is not a *braiding* (see `https://ncatlab.org/nlab/show/braided+monoidal+category`).

However, it is related to the $\star$-involution in an interesting way.

Recall that we have a unique embedding $\psi_\lambda : \mathcal{B}_\lambda \hookrightarrow \mathcal{T}_\lambda \otimes \mathcal{B}_\infty$ with $\psi(u_\lambda) = t_\lambda \otimes u_\infty$.

For $b \in \mathcal{B}_\mu$ define $b^\star = \psi_\lambda^{-1}(\psi_\mu(b)^\star) \in \mathcal{B}_\lambda$. This is well-defined in the following case:

Theorem 2.3. For a highest weight element $b \otimes u_\lambda \in \mathcal{B}_\mu \otimes \mathcal{B}_\lambda$, it holds that $C(b \otimes u_\lambda) = b^\star \otimes u_\mu$.

The proof of this result follows from some exercises in Chapter 15 of Bump and Schilling's book.

3 Crystals and tropical geometry

In the last two weeks of class we embark on the long penultimate chapter of Bump and Schilling's book.

(The final chapter of the book is just a survey of further topics.)

The first thing to discuss is the *Lusztig parametrization* of $\mathcal{B}_\infty$. This was introduced by Lusztig in the 1990s as a "canonical basis" of the quantized enveloping algebra $U_q(\mathfrak{n})$ of the maximal nilpotent Lie algebra $\mathfrak{n}$ of a semisimple Lie algebra $\mathfrak{g}$.

We will focus on the part of the Lusztig parametrization that can be explained without using quantum groups. For every reduced word $\mathbf{i} = (i_1, i_2, \dots, i_N)$ for the longest element w_0 of the Weyl group W we have a map $\mathcal{B}_\infty \to \mathbb{N}^N$, written $v \mapsto v_\mathbf{i} := \mathrm{string}_\mathbf{i}(v)$. The different maps $v \mapsto v_\mathbf{i}$ are related by piecewise-linear transformations of $\mathbb{N}^N$ which are "tropicalizations" of certain algebraic maps.

Our second topic will be the combinatorics of *MV polytopes*. As motivation, we explain now how these objects arise in the geometric Langlands program.

(In the following discussion we won't worry too much about the precise definitions.)

Let G be a split reductive group over $\mathbb{C}$ and let $\widehat{G}$ be its Langlands dual group. Write $\mathcal{O} = \mathbb{C}[[x]]$ for the ring of formal power series and $\mathcal{K} = \mathbb{C}((x))$ for its field of fractions, i.e., the field of formal Laurent series. The quotient $\mathsf{Gr} = G((\mathcal{K}))/G((\mathcal{O}))$ is called the *affine Grassmannian* or *loop Grassmannian*.

If G is a general linear group then Gr is a parameter space for the set of finitely-generated $\mathcal{O}$-submodules spanning the vector space $\mathcal{K}^n$ — hence the name.

There is an equivalence of categories, called the *geometric Satake isomorphism*, between the finite-dimensional representations of $\widehat{G}$ and the perverse sheaves on Gr.

MV polytopes are related to certain *MV-cycles*, whose images in the intersection homology of a perverse sheaf form a basis of the corresponding representation of $\widehat{G}$.

The *moment maps* of an MV-cycle represent each element of the cycle by a convex polytope in the weight lattice of $\widehat{G}$. These polytopes are organized into a crystal basis for the corresponding representation. For us, these *MV-polytopes* will provide another model for $\mathcal{B}_\infty$, in which crystal elements are polytopes. The components of $v_\mathbf{i}$ will turn out to be the lengths of the edges of one of these polytopes.

The piecewise linear maps involving max and min that we have encountered recently, and which will reappear in the following lectures, are *tropicalizations* of certain algebraic maps.

To be precise, define the *tropical semi-ring* $\mathbb{T}$ to be the set $\mathbb{R} \sqcup \{\infty\}$ with the usual addition, multiplication, and division operations replaced by

$$x \oplus y := \min(x, y), \qquad x \otimes y := x + y, \qquad \text{and} \qquad x \oslash y := x - y.$$

A *semi-ring* is defined in the same way as a ring except elements need not have additive inverses.

In other words, the additive structure of a semi-ring is a commutative monoid rather than a group.

The usual 0 element in $\mathbb{R}$ is the multiplicative identity in $\mathbb{T}$.

The element ∞ is the additive identity in $\mathbb{T}$. To indicate this, we set $\mathbb{1} = 0$ and $\mathbb{O} = \infty$.

Any polynomial map that has a formula that does not involve subtraction can be reinterpreted using the tropical relations. For example, the resulting *tropicalization* of $f(x, y, z) = (x + y)/z$ is the piecewise linear map $f(x, y, z) = \min(x, y) - z$.

Conversely, we can try to interpret a piecewise linear map as the tropicalization of a polynomial map, to be called a *geometric lifting*. Finding geometric lifts is an underdetermined problem usually without a unique solution. But it is sometimes interesting to look for such lifts.

4 The Lusztig parametrization in type A_2

We will only discuss geometric crystals for simply-laced Cartan types (Φ, Λ).

In this section we further restrict our attention to Cartan type A_2.

Let R be a commutative ring. For $a \in R$ define 3-by-3 matrices

$$x_1(a) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad x_2(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}.$$

One can check that

$$x_1(a)x_2(b)x_1(c) = \begin{bmatrix} 1 & a+c & ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad x_2(a)x_1(b)x_2(c) = \begin{bmatrix} 1 & b & bc \\ 0 & 1 & a+c \\ 0 & 0 & 1 \end{bmatrix}.$$

Define ϑ_{alg} as the map from the domain $\{(a,b,c) : a+c \neq 0,\ b \neq 0\}$ to itself given by

$$\vartheta_{\mathrm{alg}}(a,b,c) = \left(\frac{bc}{a+c}, a+c, \frac{ab}{a+c} \right).$$

Then we see that $x_1(a)x_2(b)x_1(c) = x_2(a')x_1(b')x_2(c')$ where $(a',b',c') = \vartheta_{\mathrm{alg}}(a,b,c)$.

It follows by a calculation that ϑ_{alg} has order 2.

What about the *tropicalization* of ϑ_{alg}? This is the map given by respectively replacing addition, multiplication, and division by min, addition, and subtraction; these substitutions make sense since the formula for ϑ_{alg} does not involve subtraction.

Write ϑ for the tropicalization of ϑ_{alg}. This is the piecewise linear map on $\mathbb{R}^3$ given by

$$\vartheta(a,b,c) = (b + c - \min(a,c), \min(a,c), a + b - \min(a,c)).$$

Using this map, we construct a crystal isomorphic to $\mathcal{B}_\infty$ in type A_2.

Consider the set $\mathcal{L} = \mathbb{N}^3$ of nonnegative integer vectors. Define $\mathbf{wt} : \mathbb{N}^3 \to \Lambda \cong \mathbb{Z}^3$ by

$$\mathbf{wt}(x) = -(a+b)\alpha_1 - (b+c)\alpha_2 \qquad \text{for } x = (a,b,c).$$

To be concrete, one can take $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1} \in \mathbb{Z}^3$. Next define

$$\varepsilon_1(x) = a \qquad \text{and} \qquad e_1(x) = \begin{cases} (a-1,b,c) & \text{if } a > 0 \\ 0 & \text{if } a = 0. \end{cases}$$

The other crystal operators are given by $\varepsilon_2 = \varepsilon_1 \circ \vartheta$ and $e_2 = \vartheta \circ e_1 \circ \vartheta$. Next, define

$$f_1(x) = (a+1,b,c) \qquad \text{and} \qquad f_2 = \vartheta \circ f_1 \circ \vartheta.$$

Finally let φ_i be such that $\varphi_i(x) - \varepsilon_i(x) = \langle \mathbf{wt}(x), \alpha_i^\vee \rangle$.

Proposition 4.1. The set $\mathcal{L} = \mathbb{N}^3$ relative to these operators is a type A_2 crystal with unique highest weight element $(0,0,0)$. It is a weak Stembridge crystal isomorphic to $\mathcal{B}_\infty$.

Proof. Checking that $\mathcal{L}$ is an upper seminormal A_2 crystal is a routine calculation.

Checking the Stembridge axioms takes a little more work. Let $x = (a,b,c)$. Assume $\varepsilon_1(x) = a > 0$. Then

$$\varepsilon_2(e_1(x)) = \begin{cases} \varepsilon_2(x) & \text{if } a > c \\ \varepsilon_2(x) + 1 & \text{if } a \leq c. \end{cases}$$

Replacing x by $\vartheta(x)$, this implies that

$$\varepsilon_1(e_2(x)) = \begin{cases} \varepsilon_1(x) & \text{if } a < c \\ \varepsilon_1(x) + 1 & \text{if } a \geq c. \end{cases}$$

This verifies the first two Stembridge axioms (S0) and (S1). The other axioms may be verified by explicit calculations; just plug in all the formulas. For example, one can check that

$$e_1 e_2 e_2 e_1 (a-1, b-1, a-1) = e_2 e_1 e_1 e_2 (a-1, b-1, a-1) = (a, b, a)$$

which is most of what is needed to verify the axiom (S3).

Let us conclude that $\mathcal{L}$ is a weak Stembridge crystal. The claim about the highest weight vector is obvious. Since $\mathcal{L}$ is upper seminormal with a unique highest weight vector of weight zero, it must be isomorphic to $\mathcal{B}_\infty$, since $\mathcal{B}_\infty$ is the only such crystal up to isomorphism.

(Technically, the last claim requires us to prove that upper seminormal weak Stembridge crystals are uniquely determined up to isomorphism by their highest weights. We have only discussed the version of this claim for (seminormal) Stembridge crystals. The argument for the stronger property is similar: suppose you have map that identifies highest weight elements and looks like a crystal isomorphism on the top part of the crystal graph; assume the domain of this map is maximal satisfying some natural conditions, and then argue that the domain is a proper subset of your crystal implies a contradiction.) □

5 Geometric preparations for simply-laced types

Next time we will discuss the Lusztig parametrization of $\mathcal{B}_\infty$ for simply-laced types (Φ, Λ).

This will involve generalizing the constructions for type A_2. The required piecewise linear maps will be tropicalizations of geometric ones. We discuss some of the relevant geometry here.

Not every unipotent upper triangular 3-by-3 matrix can be written as $x_1(a)x_2(b)x_1(c)$. For example,

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

for $y = 0$ and $z \neq 0$. Nevertheless set of the matrices $x_1(a)x_2(b)x_1(c)$ is Zariski dense in the group N^+ of unipotent upper triangular 3-by-3 matrices.

We can generalize this fact to an arbitrary reductive group G.

Define by $\mathbb{G}_a$ the "additive group" which is the affine algebraic group such that $\mathbb{G}_a(R)$ is the additive group of R for any commutative ring R.

If $\alpha \in \Phi$ is a root then let $x_\alpha : \mathbb{G}_a \to G$ be the one-parameter subgroup tangent to α and let $x_i = x_{\alpha_i}$.

In type A for $\alpha = \mathbf{e}_i - \mathbf{e}_j$ we have $x_\alpha(t) = I + t \cdot E_{ij}$.

Let $\mathbf{i} = (i_1, \ldots, i_N)$ be a reduced word for the longest element $w_0 \in W$. Define

$$\gamma_1(\mathbf{i}) = \alpha_{i_1}, \qquad \gamma_2(\mathbf{i}) = s_{i_1}(\alpha_{i_2}), \qquad \gamma_3(\mathbf{i}) = s_{i_1}s_{i_2}(\alpha_{i_3}), \qquad \cdots$$

and let $\gamma(\mathbf{i}) = (\gamma_1(\mathbf{i}), \ldots, \gamma_N(\mathbf{i}))$.

For example, in the type A_2 case for $\mathbf{i} = (1, 2, 1)$ we have $\gamma(\mathbf{i}) = (\alpha_1, \alpha_1 + \alpha_2, \alpha_2)$ for $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$.

The following statement is something that could be derived in a course on finite reflection groups.

Proposition 5.1. Each positive root appears exactly once in the sequence $\gamma(\mathbf{i})$.

Let V and W be irreducible algebraic varieties. A *rational map* $f : V \to W$ is a morphism defined on a dense Zariski-open subset U of V. If $f' : V \to W$ is another rational map defined on U', we consider f and f' to be the same if $f = f'$ on the (still dense) intersection $U \cap U'$.

A *birational equivalence* is a rational map that has a rational two-sided inverse, in other words, an isomorphism in the category of irreducible varieties with rational maps as morphisms.

Let N^+ be the maximal unipotent subgroup generated by the x_α with $\alpha \in \Phi^+$.

Define N^- analogously, taking $\alpha \in \Phi^-$.

Let $B = \{g \in G : gN^+g^{-1} = N^+\}$ be the normalizer of N^+ in G and let B^- be the normalizer of N^-.

Then B and B^- are opposite Borel subgroups of G.

The *flag variety* is $X = G/B^-$. This is a complete projective variety.

The *Bruhat decomposition* is the disjoint union $G = \bigsqcup_{w\in W} BwB^-$.

This induces a decomposition of X into relatively open subsets BwB^-/B^-.

There is a unique open cell BB^-/B^- corresponding to $w = 1$ and all other cells have smaller dimension.

Define $j : N^+ \to X$ to be the morphism with $j(n) = nB^-$.

This morphism is injective and its image is a dense open set. Hence j is a birational equivalence.

For $i \in I$, the subgroup $\mathrm{SL}(2)_i \subset G$ generated by x_{α_i} and $x_{-\alpha_i}$ is isomorphic to $\mathrm{SL}(2)$.

Let $B_i^- = B^- \cap \mathrm{SL}(2)_i$ and let P_i^- be the minimal parabolic subgroup generated by $\mathrm{SL}(2)_i$ and B^-.

The group $(B^-)^N$ acts on the product $P_{i_1}^- \times \cdots \times P_{i_N}^-$ on the right by

$$(b_1, \dots, b_N) : (p_1, \dots, p_N) \mapsto (p_1b_1,\ b_1^{-1}p_2b_2,\ \dots,\ b_{N-1}^{-1}p_Nb_N).$$

The *Bott-Samelson variety* $X_\mathbf{i}$ is the quotient of $P_{i_1}^- \times \cdots \times P_{i_N}^-$ by this action.

This is a projective variety of dimension N.

There is a morphism $\pi_\mathbf{i} : X_\mathbf{i} \to X$ sending the orbit of $(p_1, \dots, p_N)$ to the coset $p_1 \cdots p_N B^-$.

Proposition 5.2. The map $\pi_\mathbf{i}$ is a birational equivalence.

Proof. The image of BB^-/B in X is an affine space that may be identified with N^+ since the map $n \mapsto nB^-$ from N^+ to BB^-/B is an isomorphism.

Let $\mathbb{A}^N$ be affine N-space. We may map $\mathbb{A}^N$ into $X_\mathbf{i}$ by sending $a = (a_1, \dots, a_N) \in \mathbb{A}^N$ to the orbit of

$$(x_{i_1}(a_1)s_{i_1}, \dots, x_{i_N}(a_N)s_{i_N}) \in \mathrm{SL}(2)_{i_1} \times \cdots \times \mathrm{SL}(2)_{i_N}.$$

The elements of this form make up an open subset U or $X_\mathbf{i}$.

Under $\pi_\mathbf{i}$ the above element maps to $\xi_1 \cdots \xi_N$ where $\xi_k = (s_{i_1} \cdots s_{i_{k-1}})x_{i_k}(a_k)(s_{i_1} \cdots s_{i_{k-1}})^{-1} \in x_{\gamma_k}(\mathbb{G}_a)$. Since the γ_k are the positive roots of G, every element of N^+ may be written uniquely in this form. Hence $\pi_\mathbf{i}$ maps the open set U in $X_\mathbf{i}$ bijectively onto the big cell of X, so is a birational equivalence. □

Now define a morphism $\phi_\mathbf{i} : \mathbb{A}^N \to X$ by $\phi_i(a_1, \dots, a_N) = x_{i_1}(a_1) \cdots x_{i_N}(a_N)B^-$.

Theorem 5.3. The morphism $\phi_\mathbf{i} : \mathbb{A}^N \to X$ is a birational equivalence.

Proof. Let $\widetilde{\phi}_\mathbf{i} : \mathbb{A}^N \to X_\mathbf{i}$ be the map sending $(a_1, \dots, a_N)$ to the orbit of $(x_{i_1}(a_1), \dots, x_{i_N}(a_N))$. This map lifts $\phi_\mathbf{i}$ to be the Bott-Samelson variety since $\pi_\mathbf{i} \circ \widetilde{\phi}_\mathbf{i} = \phi_\mathbf{i}$. Now $\widetilde{\phi}_\mathbf{i}$ is a birational morphism, essentially since the inclusion of $\mathbb{A}^1$ into $\mathbb{P}^1$ is birational. Since $\pi_\mathbf{i}$ is a birational equivalence, the theorem follows. □