1 Last time: some (tropical) geometry

The *tropical semi-ring* $\mathbb{T}$ is the set $\mathbb{R} \sqcup \{\infty\}$ with the operations $+$, $\times$, and $/$ replaced by

$$x \oplus y := \min(x, y), \qquad x \otimes y := x + y, \qquad \text{and} \qquad x \oslash y := x - y.$$

We write $\mathbb{1} = 0$ and $\mathbb{0} = \infty$ since these are the identities for $\oplus$ and $\otimes$.

The *tropicalization* of a polynomial that does not involve subtraction is the piecewise linear map given by replacing $+$, $\times$, $/$ with $\oplus$, $\otimes$, $\oslash$. Example: $f(x, y, z) = (x + y)/z$ becomes $f(x, y, z) = \min(x, y) - z$.

Let $G = \mathrm{GL}(n, \mathbb{C})$ and let $\mathbb{G}_a = (\mathbb{C}, +)$ be the additive group of $\mathbb{C}$.

Given $\alpha \in \Phi = \{\mathbf{e}_i - \mathbf{e}_j : 1 \leq i, j \leq n,\ i \neq j\}$ define $x_\alpha(r) = I + rE_{ij} \in G$ and $x_i := x_{\alpha_i} = x_{\mathbf{e}_i - \mathbf{e}_{i+1}}$.

Let $\mathbf{i} = (i_1, \ldots, i_N)$ be a reduced word for the longest element $w_0 \in W = S_n$.

Nontrivial fact: each positive root appears exactly once in the sequence $\gamma(\mathbf{i}) = (\gamma_1(\mathbf{i}), \ldots, \gamma_N(\mathbf{i}))$ where

$$\gamma_1(\mathbf{i}) = \alpha_{i_1}, \qquad \gamma_2(\mathbf{i}) = s_{i_1}(\alpha_{i_2}), \qquad \gamma_3(\mathbf{i}) = s_{i_1}s_{i_2}(\alpha_{i_3}), \qquad \cdots$$

Let V and W be irreducible algebraic varieties. A *rational map* $f : V \to W$ is a morphism defined on a dense Zariski-open subset U of V. A *birational equivalence* is a rational map with a rational two-sided inverse.

Let $\mathcal{N}^+ = \langle x_\alpha(r) : \alpha \in \Phi^+,\ r \in \mathbb{C}\rangle$ be the unipotent subgroup of upper-triangular matrices in G with all diagonal entries equal to one. Define $\mathcal{N}^- = \langle x_\alpha(r) : \alpha \in \Phi^-,\ r \in \mathbb{C}\rangle = \{g^T : g \in \mathcal{N}^+\}$.

Let B be the normalizer of $\mathcal{N}^+$ in G and let B^- be the normalizer of $\mathcal{N}^-$.

You can check that B (or B^-) is the subgroup of all invertible upper (or lower) triangular matrices in G.

One calls B and B^- opposite *Borel subgroups* of G. The *flag variety* is the set of cosets $X := G/B^-$.

The *Bruhat decomposition* is the disjoint union $G = \bigsqcup_{w \in W} BwB^-$.

Define $j : \mathcal{N}^+ \to X$ to be the morphism with $j(n) = nB^-$. The map j is a birational equivalence.

For $i \in \{1, 2, \ldots, n-1\}$, the subgroup $\mathrm{SL}(2)_i := \langle x_{\pm\alpha_i}(r) : r \in \mathbb{C}\rangle \subset G$ is isomorphic to $\mathrm{SL}(2)$.

Let $B_i^- = B^- \cap \mathrm{SL}(2)_i$ and let P_i^- be the minimal parabolic subgroup generated by $\mathrm{SL}(2)_i$ and B^-.

In the concrete $\mathrm{GL}(n, \mathbb{C})$ case, the elements of P_i^- are the matrices of the form

$$\begin{bmatrix} X & 0 & 0 & 0 \\ u^T & a & b & 0 \\ v^T & c & d & 0 \\ A & u' & v' & X' \end{bmatrix}$$

where $a, b, c, d \in \mathbb{C}$ have $ad - bc = 1$, where X and X' are invertible and lower triangular of size $(i-1) \times (i-1)$ and $(n-i-1) \times (n-i-1)$, respectively, and where $u, v \in \mathbb{C}^{i-1}$ and $u', v' \in \mathbb{C}^{n-i-1}$ and A is an arbitrary matrix. The product group $(B^-)^N$ acts on $P_{i_1}^- \times \cdots \times P_{i_N}^-$ on the right by

$$(b_1, \ldots, b_N) : (p_1, \ldots, p_N) \mapsto (p_1 b_1,\ b_1^{-1} p_2 b_2,\ \ldots,\ b_{N-1}^{-1} p_N b_N).$$

The *Bott-Samelson variety* $X_{\mathbf{i}}$ is the quotient of $P_{i_1}^- \times \cdots \times P_{i_N}^-$ by this action.

Let $\pi_{\mathbf{i}} : X_{\mathbf{i}} \to X$ be the morphism sending the orbit of $(p_1, \ldots, p_N)$ to the coset $p_1 \cdots p_N B^-$.

Let $\phi_{\mathbf{i}} : \mathbb{C}^N \to X$ be the morphism $\phi_i(a_1, \ldots, a_N) = x_{i_1}(a_1) \cdots x_{i_N}(a_N)B^-$.

Then both $\pi_{\mathbf{i}}$ and $\phi_{\mathbf{i}}$ are birational equivalences.

All of the above facts hold for an arbitrary reductive group G with root system Φ and Weyl group W.

2 Lusztig's parametrization of $\mathcal{B}_\infty$ for simply-laced types

Everything in this section works for arbitrary simply-laced types, but to be concrete it may be helpful to assume that (Φ, Λ) is the $\mathrm{GL}(n)$ Cartan type and that $W = S_n$.

Let $\mathrm{Red}(w)$ be the set of reduced words for $w \in W$.

Recall that the set $\mathrm{Red}(w)$ is connected by the braid relations

(B1) $\mathbf{i} = (\cdots, a, b, \cdots) \leftrightarrow (\cdots, b, a, \cdots) = \mathbf{j}$ if $s_a s_b \in W$ has order 2, along with

(B2) $\mathbf{i} = (\cdots, a, b, a, \cdots) \leftrightarrow (\cdots, b, a, b, \cdots) = \mathbf{j}$ if $s_a s_b \in W$ has order 3.

Lusztig's parametrization of $\mathcal{B}_\infty$ depends on a family of piecewise linear maps indexed by pairs of elements of $\mathrm{Red}(w)$. The description of $\mathcal{B}_\infty$ we covered a few weeks back, due to Kashiwara, may be understood in a similar way. This will be helpful to motivate Lusztig's construction.

Embed $\mathcal{B}_\infty$ into $\mathcal{B}_{i_1} \otimes \cdots \otimes \mathcal{B}_{i_N}$ in the usual way, where $\mathbf{i} = (i_1, i_2, \dots, i_N) \in \mathrm{Red}(w_0)$.

Let $u_{i_1}(-a_1) \otimes \cdots \otimes u_{i_N}(-a_N)$ be the image of $v \in \mathcal{B}_\infty$ under this embedding and define $\widehat{v}_{\mathbf{i}} = (a_1, \dots, a_N)$.

The map $v \mapsto \widehat{v}_{\mathbf{i}} \in \mathbb{Z}^N$ gives us a "view" of the crystal $\mathcal{B}_\infty$.

A relevant question is how such views vary for different choices of $\mathbf{i}$.

To answer this question, we make use of two maps $\theta_2 : \mathbb{Z}^2 \to \mathbb{Z}^2$ and $\theta_3 : \mathbb{Z}^3 \to \mathbb{Z}^3$ defined by

$$\theta_2(a, b) = (b, a) \qquad \text{and} \qquad \theta_3(a, b, c) = (\max(c, b - a), a + c, \min(b - c, a)).$$

We have encountered θ_3 a few times already; it may also be expressed as

$$\theta_3(a, b, c) = (b + c - \min(b, a + c), a + c, \min(b - c, a))$$

which is the tropicalization of the algebraic map $(a, b, c) \mapsto \left(\frac{bc}{b+ac}, ac, \frac{b+ac}{c}\right)$.

Proposition 2.1. Let $\mathbf{i}, \mathbf{j} \in \mathrm{Red}(w_0)$. Then there is a piecewise linear map $\widehat{\mathcal{R}}_{\mathbf{i},\mathbf{j}} : \mathbb{Z}^N \to \mathbb{Z}^N$ such that if $v \in \mathcal{B}_\infty$ then $\widehat{\mathcal{R}}_{\mathbf{i},\mathbf{j}}(\widehat{v}_{\mathbf{i}}) = \widehat{v}_{\mathbf{j}}$. These maps satisfy the relation

$$\widehat{\mathcal{R}}_{\mathbf{j},\mathbf{k}} \circ \widehat{\mathcal{R}}_{\mathbf{i},\mathbf{j}} = \widehat{\mathcal{R}}_{\mathbf{i},\mathbf{k}}.$$

If $\mathbf{i}$ and $\mathbf{j}$ are as in (B1) for $(i_k, i_{k+1}) = (a, b)$ then $\widehat{\mathcal{R}}_{\mathbf{i},\mathbf{j}}$ is θ_2 applied to the pair of consecutive positions (a_k, a_{k+1}) in $\widehat{v}_{\mathbf{i}} = (a_1, \dots, a_N)$, and if $\mathbf{i}$ and $\mathbf{j}$ are as in (B2) for $(i_k, i_{k+1}, i_{k+2}) = (a, b, c)$ then $\widehat{\mathcal{R}}_{\mathbf{i},\mathbf{j}}$ is θ_3 applied to the triple of consecutive positions (a_k, a_{k+1}, a_{k+2}) in $\widehat{v}_{\mathbf{i}}$.

Proof. This follows from the fact that the maps θ_2 and θ_3 are isomorphisms $\mathcal{B}_i \otimes \mathcal{B}_j \to \mathcal{B}_j \otimes \mathcal{B}_i$ and $\mathcal{B}_i \otimes \mathcal{B}_{i+1} \otimes \mathcal{B}_i \to \mathcal{B}_{i+1} \otimes \mathcal{B}_i \otimes \mathcal{B}_{i+1}$, which we showed in Proposition 5.1 of Lecture 17. □

The Lusztig parametrization will rely on slightly different piecewise linear maps $\mathcal{R}_{\mathbf{i},\mathbf{j}} : \mathbb{Z}^N \to \mathbb{Z}^N$ (note the change of notation) with similar formal properties.

From last time, let $\vartheta : \mathbb{R}^3 \to \mathbb{R}^3$ be the piecewise linear map given by

$$\vartheta(a, b, c) = (b + c - \min(a, c), \min(a, c), a + b - \min(a, c)).$$

Recall that this is the tropicalization of the map

$$\vartheta_{\mathrm{alg}}(a, b, c) = \left(\frac{bc}{a + c}, a + c, \frac{ab}{a + c}\right).$$

Proposition 2.2. There exists a family of piecewise linear maps $\mathcal{R}_{\mathbf{i},\mathbf{j}} : \mathbb{Z}^N \to \mathbb{Z}^N$ satisfying

$$\mathcal{R}_{\mathbf{j},\mathbf{k}} \circ \mathcal{R}_{\mathbf{i},\mathbf{j}} = \mathcal{R}_{\mathbf{i},\mathbf{k}}$$

along with the following properties: If $\mathbf{i}$ and $\mathbf{j}$ are as in (B1) for $(i_k, i_{k+1}) = (a, b)$ then $\mathcal{R}_{\mathbf{i},\mathbf{j}}$ is θ_2 applied to the substring (a_k, a_{k+1}) in $(a_1, \dots, a_N)$. If $\mathbf{i}$ and $\mathbf{j}$ are as in (B2) for $(i_k, i_{k+1}, i_{k+2}) = (a, b, c)$ then $\mathcal{R}_{\mathbf{i},\mathbf{j}}$ is ϑ applied to the substring (a_k, a_{k+1}, a_{k+2}) in $(a_1, \dots, a_N)$.

Proof. Results from last time show that there are birational maps $\mathfrak{R}_{\mathbf{i},\mathbf{j}} : \mathbb{C}^N \to \mathbb{C}^N$ with $\mathfrak{R}_{\mathbf{j},\mathbf{k}} \circ \mathfrak{R}_{\mathbf{i},\mathbf{j}} = \mathfrak{R}_{\mathbf{i},\mathbf{k}}$ such that in case (B1) the map $\mathfrak{R}_{\mathbf{i},\mathbf{j}}$ when applied to $(a_1, \dots, a_N)$ interchanges a_k and a_{k+1}, while in case (B2) the same map applies ϑ_{alg} to (a_k, a_{k+1}, a_{k+2}). Namely, define

$$\mathfrak{R}_{\mathbf{i},\mathbf{j}} = \phi_{\mathbf{j}}^{-1} \phi_{\mathbf{i}}$$

where $\phi_{\mathbf{i}}$ is our birational equivalence $\mathbb{C}^N \to X$ to the flag variety.

The property we want in case (B1) holds automatically and what needs to happen in case (B2) follows from our discussion of the Lusztig parametrization in type A_2, namely, the identity

$$x_1(a)x_2(b)x_1(c) = x_2(a')x_1(b')x_2(c') \qquad \text{where } (a', b', c') = \vartheta_{\mathrm{alg}}(a, b, c).$$

The desired maps $\mathcal{R}_{\mathbf{i},\mathbf{j}}$ are given as the tropicalizations of $\mathfrak{R}_{\mathbf{i},\mathbf{j}}$. □

We may now define a crystal $\mathcal{L}$ as follows. An element v of $\mathcal{L}$ consists of a family of "views" $v = (v_{\mathbf{i}})$ indexed by $\mathbf{i} \in \mathrm{Red}(w_0)$. Views are vectors $v_{\mathbf{i}} \in \mathbb{N}^N$, which must be related by the rule $v_{\mathbf{j}} = \mathcal{R}_{\mathbf{i},\mathbf{j}}(v_{\mathbf{i}})$. The latter property implies that the entire family v is determined by any particular view $v_{\mathbf{i}}$.

Given such a family $v = (v_{\mathbf{i}}) \in \mathcal{L}$, define

$$\mathbf{wt}(v_{\mathbf{i}}) = -\sum_{j=1}^{N} a_j \gamma_j$$

where $v_{\mathbf{i}} = (a_1, a_2, \dots, a_N)$ and where $\gamma(\mathbf{i}) = (\gamma_1, \gamma_2, \dots, \gamma_N)$ is the sequence of positive roots

$$\gamma_1 = \alpha_{i_1}, \qquad \gamma_2 = s_{i_1}(\alpha_{i_2}), \qquad \gamma_3 = s_{i_1} s_{i_2}(\alpha_{i_3}), \qquad \cdots$$

with $\{\alpha_i : i \in I\}$ the set of simple roots and $s_i = r_{\alpha_i} \in W$. Recall that $\{\gamma_1, \gamma_2, \dots, \gamma_N\} = \Phi^+$.

Lemma 2.3. If $v = (v_{\mathbf{i}}) \in \mathcal{L}$ then the value of $\mathbf{wt}(v_{\mathbf{i}})$ is the same for all $i \in \mathrm{Red}(w_0)$.

Proof. Fix $v = (v_{\mathbf{i}}) \in \mathcal{L}$ and suppose $\mathbf{i}, \mathbf{j} \in \mathrm{Red}(w_0)$ are reduced words related by either (B1) or (B2).

It suffices to show that $\mathbf{wt}(v_{\mathbf{i}}) = \mathbf{wt}(v_{\mathbf{j}})$.

In the (B1) case, the view $v_{\mathbf{j}}$ is obtained by interchanging two adjacent entries in $v_{\mathbf{i}}$, say a_k and a_{k+1}, and we claim that $\gamma(\mathbf{j})$ is likewise formed by interchanging γ_k and γ_{k+1} in $\gamma(\mathbf{i})$. To see that this holds, write $w = s_{i_1} \cdots s_{i_{k-1}}$. Then the roots in $\gamma(\mathbf{i})$ have the formula $\gamma_k = w(\alpha_k)$ and $\gamma_{k+1} = w s_{i_k}(\alpha_{i_{k+1}})$. Since s_{i_k} and $s_{i_{k+1}}$ commute, the roots α_{i_k} and $\alpha_{i_{k+1}}$ are orthogonal, so $s_{i_k}(\alpha_{i_{k+1}}) = \alpha_{i_{k+1}}$. Thus

$$(\gamma_k, \gamma_{k+1}) = (w\alpha_{i_k}, w\alpha_{i_{k+1}}).$$

The same argument shows that kth and $(k+1)$th roots in $\gamma(\mathbf{j})$ are $(w\alpha_{i_{k+1}}, w\alpha_{i_k})$, so our claim holds.

We conclude in the (B1) case that $\mathbf{wt}(v_{\mathbf{i}}) = \mathbf{wt}(v_{\mathbf{j}})$. Now assume $\mathbf{i}$ and $\mathbf{j}$ are connected by relation (B2), say in positions k, $k+1$, $k+2$. Then in the sequence $\gamma(\mathbf{i}) = (\gamma_1, \gamma_2, \dots, \gamma_N)$ we have

$$(\gamma_k, \gamma_{k+1}, \gamma_{k+2}) = (w(\alpha_{i_k}), w s_{i_k}(\alpha_{i_{k+1}}), w s_{i_k} s_{i_{k+1}}(\alpha_{i_k})).$$

Since $s_{i_k}s_{i_{k+1}} \in W$ has order 3, we have $s_{i_k}(\alpha_{i_{k+1}}) = \alpha_{i_k} + \alpha_{i_{k+1}}$ and $s_{i_k}s_{i_{k+1}}(\alpha_{i_k}) = \alpha_{i_{k+1}}$, so

$$(\gamma_k, \gamma_{k+1}, \gamma_{k+2}) = (w(\alpha_{i_k}), w(\alpha_{i_k} + \alpha_{i_{k+1}}), w(\alpha_{i_{k+1}})).$$

By the same argument, the corresponding terms in $\gamma(\mathbf{j})$ are $(w(\alpha_{i_{k+1}}), w(\alpha_{i_k} + \alpha_{i_{k+1}}), w(\alpha_{i_k}))$. Thus

$$\begin{aligned}\mathbf{wt}(v_{\mathbf{i}}) - \mathbf{wt}(v_{\mathbf{j}}) &= (a_k + a_{k+1} - \min(a_k, a_{k+2}) - a_k)w(\alpha_{i_k}) \\ &+ (\min(a_k, a_{k+2}) - a_{k+1})w(\alpha_{i_k} + \alpha_{i_{k+1}}) \\ &+ (a_{k+1} + a_{k+2} - \min(a_k, a_{k+2}) - a_{k+2})w(\alpha_{i_{k+1}}) = 0\end{aligned}$$

since the terms of $v_{\mathbf{j}}$ in positions $k, k+1, k+2$ are $\vartheta(a_k, a_{k+1}, a_{k+2})$. □

We produce a crystal structure on $\mathcal{L}$ as follows.

The weight map is $\mathbf{wt}(v) = \mathbf{wt}(v_{\mathbf{i}})$ for $v = (v_{\mathbf{i}}) \in \mathcal{L}$. This is well-defined by the preceding lemma.

To define $e_i(v)$, $f_i(v)$, $\varepsilon_i(v)$, $\varphi_i(v)$, choose a word $\mathbf{i} \in \mathrm{Red}(w_0)$ such that $i_1 = i$.

Then let $\varepsilon_i(v) = a_1$ where $v_{\mathbf{i}} = (a_1, \dots, a_N)$. We define $e_i(v) = 0$ if $a_1 = 0$.

If $a_1 \neq 0$ then $e_i(v) = v'$ where $v' \in \mathcal{L}$ is the unique element with

$$v'_{\mathbf{i}} = (a_1 - 1, a_2, \dots, a_N).$$

We define $\varphi_i(v)$ by requiring that $\varphi_i(v) - \varepsilon_i(v) = \langle \mathbf{wt}(v), \alpha_i^\vee \rangle$.

Finally, f_i increments a_1 in $v_{\mathbf{i}}$ without affecting any of the other a_k, i.e., $f_i(v) = v'' \in \mathcal{L}$ where

$$v''_{\mathbf{i}} = (a_1 + 1, a_2, \dots, a_N).$$

Lemma 2.4. These definitions of $e_i(v)$, $f_i(v)$, $\varepsilon_i(v)$, $\varphi_i(v)$ are independent of the choice of $\mathbf{i}$.

Proof. If $\mathbf{j} \in \mathrm{Red}(w_0)$ also has first entry $j_1 = i$, then by applying Matsumoto's theorem to the word $s_i^{-1}w_0$ we obtain a sequence of reduced words interpolating between $(i_2, \dot{;} i_k)$ and $(j_2, \dots, j_k)$ in which consecutive words are related as in (B1) or (B2).

Prepending i to each of these words gives a sequence of reduced words $\mathbf{i} = \mathbf{i}_0, \mathbf{i}_1, \dots, \mathbf{i}_r = \mathbf{j}$, all beginning with i, in which consecutive words are related as in (B1) or (B2).

The composition $\mathcal{R}_{\mathbf{i},\mathbf{j}} = \mathcal{R}_{\mathbf{i}_{r-1},\mathbf{i}_r} \circ \cdots \circ \mathcal{R}_{\mathbf{i}_1,\mathbf{i}_2}$ therefore does not affect the first coordinate, so the first entry a_1 in $v_{\mathbf{i}}$ is the same as in $v_{\mathbf{j}}$, and this completely determines $e_i(v)$, $f_i(v)$, $\varepsilon_i(v)$, $\varphi_i(v)$. □

In type A_2, there are two reduced words $\mathbf{i} = (1, 2, 1)$ and $\mathbf{j} = (2, 1, 2)$.

If $v = (a, b, c)$ is a vector in $\mathbb{N}^3$, which is what we called $\mathcal{L}$ last time, then we define the two views of v to be $v_{\mathbf{i}} = (a, b, c)$ and $v_{\mathbf{j}} = \vartheta(a, b, c)$.

This change of notation $v = (a, b, c) \mapsto v = (v_{\mathbf{i}})$ identifies the type A_2 crystal $\mathcal{L}$ defined last time as a subset of $\mathbb{N}^3$ with our new crystal $\mathcal{L}$ defined as a subset of $(\mathbb{N}^3)^2 = (\mathbb{N}^{n+1})^{|\mathrm{Red}(w_0)|}$ for $n = 2$.

Theorem 2.5. In any simply-laced type, the crystal $\mathcal{L}$ is isomorphic to $\mathcal{B}_\infty$.

Proof sketch. First, we want to check that $\mathcal{L}$ is weakly Stembridge, i.e., satisfies all of the Stembridge axioms except the requirement of being seminormal.

The trivial axioms (S0) and (S0′) hold since $\mathcal{L}$ is upper seminormal and $f_i(v)$ is never zero.

The remaining Stembridge axioms require us to work with crystal operators for a pair of indices $i, j \in I$. Similar to the previous lemma, we can always choose a view $v_{\mathbf{i}}$ indexed by a reduced word $\mathbf{i} \in \mathrm{Red}(w_0)$

that begins as $(i, j, \dots)$ or $(i, j, i, \dots)$. This lets us calculate all of the relevant crystal operators explicitly. The details are the same as what we sketched last time for the Lusztig parametrization in type A_2.

Once we are satisfied that $\mathcal{L}$ is a weakly Stembridge crystal, we may deduce that there exists a unique crystal embedding $\tau : \mathcal{L} \hookrightarrow \mathcal{B}_\infty$, since $\mathcal{L}$ is connected by construction with unique highest element given by the family of "zero views" $v = (v_{\mathbf{i}})$ in which $v_{\mathbf{i}} = (0, 0, \dots, 0)$ for all $\mathbf{i} \in \mathrm{Red}(w_0)$.

We want to show that τ is surjective. Argue by contradiction. Suppose $x \in \mathcal{B}_\infty$ is maximal under $\prec$ among all elements not in the image of τ. This element cannot be the unique highest weight element, which is in the image of τ. Therefore $x = f_i(y)$ for some $y = \tau(y')$ where $y' \in \mathcal{L}$. But by definition $f_i(y')$ is never zero, so $\tau(f_i(y')) = f_i(\tau(y')) = f_i(y) = x$, which is a contradiction. Thus τ is an isomorphism. $\square$

A *root partition* of a weight μ is a tuple $(k_\alpha)_{\alpha \in \Phi^+}$ in which $k_\alpha \in \mathbb{N}$ and $\sum_{\alpha \in \Phi^+} k_\alpha \alpha = \mu$.

The *Kostant partition function* $P(\mu)$ computes the number of root partitions of μ.

One has the following identity of generating functions:

$$\prod_{\alpha \in \Phi^+} (1 - t^{-\alpha})^{-1} = \sum_{\mu \in \Lambda} P(\mu) t^{-\mu}.$$

This turns out to be the character of $\mathcal{B}_\infty$. (Showing this for Cartan type $\mathrm{GL}(n)$ is an exercise in HW5.) Deriving this from our first construction of $\mathcal{B}_\infty$ — the so-called the Kashiwara approach — is not straightforward. An advantage of the Lusztig parametrization is that it makes it obvious that this generating function is the character of $\mathcal{L} \cong \mathcal{B}_\infty$.

Corollary 2.6. One has $\mathrm{ch}(\mathcal{L}) = \mathrm{ch}(\mathcal{B}_\infty) = \sum_{\mu \in \Lambda} P(\mu) t^{-\mu}$.

Proof. The value of $\mathbf{wt}(v)$ for $v \in \mathcal{L}$ varies over all linear combinations

$$-a_1\gamma_1 - a_2\gamma_2 - \cdots - a_N\gamma_N$$

with $\Phi^+ = \{\gamma_1, \dots, \gamma_N\}$ and $a_i \in \mathbb{N}$, so $\mathrm{ch}(\mathcal{L})$ is the linear combination over all dominant weights μ of the formal elements $t^{-\mu}$ with coefficient $P(\mu)$. $\square$

3 Weyl group action

Let $\mathcal{B}$ be a seminormal crystal. Recall that for each $i \in I$ we have a map $\sigma_i : \mathcal{B} \to \mathcal{B}$ given by

$$\sigma_i(x) = \begin{cases} f_i^k(x) & \text{if } k \geq 0 \\ e_i^{-k}(x) & \text{if } k < 0 \end{cases} \qquad \text{where } k = \langle \mathbf{wt}(x), \alpha_i^\vee \rangle.$$

Moreover, if $\mathcal{B}$ is normal then there is a unique action of W on $\mathcal{B}$ in which s_i operates as σ_i.

We can apply the above definition of σ_i on other crystals. For this to define a W-action, the formula for $\sigma_i(x)$ must always give another element of $\mathcal{B}$, never zero. When $\mathcal{B}$ is not seminormal this property can fail, and indeed, this definition does not work to define a Weyl group action on $\mathcal{L}$.

However, one can use a variant $\widehat{\mathcal{L}}$ of $\mathcal{L}$ in which e_i and f_i never act as zero.

To construct this, recall that elements of $\mathcal{L}$ are families of views $v = (v_{\mathbf{i}})$, indexed by $\mathbf{i} \in \mathrm{Red}(w_0)$ with $v_{\mathbf{i}} \in \mathbb{N}^n$, satisfying $v_{\mathbf{j}} = \mathcal{R}_{\mathbf{i},\mathbf{j}}(v_{\mathbf{i}})$ for all $\mathbf{i}, \mathbf{j} \in \mathrm{Red}(w_0)$.

We define $\widehat{\mathcal{L}}$ in the same way, except now we allow views to be vectors $v_{\mathbf{i}} \in \mathbb{Z}^n$.

In $\mathcal{L}$ we defined $e_i(v) = 0$ if $\varepsilon_i(v) = 0$; in $\widehat{\mathcal{L}}$ we modify this so that $e_i(v)$ is never zero.

Explicitly, if $\mathbf{i} \in \mathrm{Red}(w_0)$ has $i = \mathbf{i}_1$, and $v \in \widehat{\mathcal{L}}$ is such that

$$v_{\mathbf{i}} = (a_1, a_2, \dots, a_N),$$

then we define $e_i(v) = v'$ to be the unique family $v' \in \widehat{\mathcal{L}}$ with $v'_{\mathbf{i}} = (a_1 - 1, a_2, \dots, a_N)$.

We retain from $\mathcal{L}$ the meaning of all other operators $\mathbf{wt}$, ε_i, f_i, φ_i on $\widehat{\mathcal{L}}$.

The same proofs go through to show that $\widehat{\mathcal{L}}$ is a crystal.

Now our formula for σ_i is never zero on $\widehat{\mathcal{L}}$. We generalize this definition slightly. Given $\nu \in \Lambda$, let

$$\sigma_i^\nu(x) = \begin{cases} f_i^k(x) & \text{if } k \geq 0 \\ e_i^{-k}(x) & \text{if } k < 0 \end{cases} \qquad \text{where } k = \langle \mathbf{wt}(x) + \nu, \alpha_i^\vee \rangle.$$

Let $\mathcal{T}_\nu = \{t_\nu\}$ be our usual 1-element crystal. Then $\sigma_i(t_\nu \otimes v) = t_\nu \otimes \sigma_i^\nu(v)$.

Theorem 3.1. Assume (as we have done throughout) that our Cartan type (Φ, Λ) is simply-laced. Then the maps $\sigma_i : \mathcal{T}_\nu \otimes \widehat{\mathcal{L}} \to \mathcal{T}_\nu \otimes \widehat{\mathcal{L}}$ are involutions satisfying the braid relations of the Weyl group W. Hence, there is a unique action of W on $\mathcal{T}_\nu \otimes \widehat{\mathcal{L}}$ in which s_i acts as σ_i.

Proof sketch. We give a proof for Cartan type A_2, since the general simply-laced case reduces to this one.

Showing that σ_i and σ_j commute if $\langle \alpha_i, \alpha_j^\vee \rangle = 0$ is relatively straightforward.

The hard part to check is that $\sigma_1^\nu \sigma_2^\nu \sigma_1^\nu = \sigma_2^\nu \sigma_1^\nu \sigma_2^\nu$ if $\langle \alpha_1, \alpha_2^\vee \rangle = -1$.

Let $\mathbb{F}$ be a field with a nonzero element M. Define $s^M_{\mathrm{alg}} : \mathbb{F}^3 \to \mathbb{F}^3$ by

$$s^M_{\mathrm{alg}}(a, b, c) = (\tfrac{Mc}{ab}, b, c).$$

With ϑ_{alg} defined as above, it holds that

$$\vartheta_{\mathrm{alg}} \circ s^{M_2}_{\mathrm{alg}} \circ \vartheta_{\mathrm{alg}} \circ s^{M_1}_{\mathrm{alg}} \circ \vartheta_{\mathrm{alg}} \circ s^{M_2}_{\mathrm{alg}} = s^{M_1}_{\mathrm{alg}} \circ \vartheta_{\mathrm{alg}} \circ s^{M_2}_{\mathrm{alg}} \circ \vartheta_{\mathrm{alg}} \circ s^{M_1}_{\mathrm{alg}} \circ \vartheta_{\mathrm{alg}}.$$

This identity follows by checking that both sides give the map $(a, b, c) \mapsto (a', b', c')$ where

$$a' = \tfrac{bcM_2 + M_1M_2}{a^2b + abc + aM_2 + cM_2}, \qquad b' = \tfrac{a^2bM_1 + abcM_1 + aM_1M_2 + cM_1M_2}{ab^2c + abM_1}, \qquad c' = \tfrac{abcM_2 + aM_1M_2}{a^2bc + abc^2 + acM_2 + c^2M_2}.$$

If s^M and ϑ are the tropicalizations of s^M_{alg} and ϑ_{alg}, then we have

$$s^M(a, b, c) = (M + c - a - b, b, c) \quad \text{and} \quad \vartheta(a, b, c) = (b + c - \min(a, c), \min(a, c), a + b - \min(b, c))$$

and it follows that

$$(\vartheta \circ s^{M_2} \circ \vartheta) \circ s^{M_1} \circ (\vartheta \circ s^{M_2} \circ \vartheta) = s^{M_1} \circ (\vartheta \circ s^{M_2} \circ \vartheta) \circ s^{M_1}.$$

Write $\alpha_i^\vee = \mathbf{e}_i - \mathbf{e}_{i+1}$. Recall from last time that the weight map $\widehat{\mathcal{L}}$ in type A_2 has the formula

$$\mathbf{wt}(v) = \begin{cases} (-a - b, a - c, b + c) & \text{if } v_{(1,2,1)} = (a, b, c) \\ (-b - c, c - a, a + b) & \text{if } v_{(2,1,2)} = (a, b, c). \end{cases}$$

Thus, if $M_1 = \langle \nu, \alpha_1^\vee \rangle$ and $M_2 = \langle \nu, \alpha_2^\vee \rangle$ then for $\mathbf{i} = (1, 2, 1)$ and $\mathbf{j} = (2, 1, 2)$ we have

$$(\sigma_1^\nu v)_{\mathbf{i}} = s^{M_1}(v_{\mathbf{i}}), \qquad (\sigma_2^\nu v)_{\mathbf{j}} = s^{M_2}(v_{\mathbf{j}}), \qquad v_{\mathbf{j}} = \vartheta(v_{\mathbf{i}}).$$

Putting these together gives $(\sigma_2^\nu v)_{\mathbf{i}} = \vartheta \circ s^{M_2} \circ \vartheta(v_{\mathbf{i}})$, so

$$\begin{aligned} (\sigma_2^\nu \sigma_1^\nu \sigma_2^\nu v)_{\mathbf{i}} &= (\vartheta \circ s^{M_2} \circ \vartheta) \circ s^{M_1} \circ (\vartheta \circ s^{M_2} \circ \vartheta)(v_{\mathbf{i}}) = s^{M_1} \circ (\vartheta \circ s^{M_2} \circ \vartheta) \circ s^{M_1}(v_{\mathbf{i}}) \\ &= (\sigma_1^\nu \sigma_2^\nu \sigma_1^\nu v)_{\mathbf{i}} \end{aligned}$$

as needed. □