1 Last time: Lusztig's parametrization for simply-laced types

Let (Φ, Λ) be a simply-laced Cartan type with simple roots $\{\alpha_i : i \in I\}$ and Weyl group $W = \langle s_i : i \in I\rangle$.

The set $\mathrm{Red}(w)$ of reduced words for any $w \in W$ is connected by the braid relations

(B1) $\mathbf{i} = (\cdots, a, b, \cdots) \leftrightarrow (\cdots, b, a, \cdots) = \mathbf{j}$ if $s_a s_b \in W$ has order 2, along with

(B2) $\mathbf{i} = (\cdots, a, b, a, \cdots) \leftrightarrow (\cdots, b, a, b, \cdots) = \mathbf{j}$ if $s_a s_b \in W$ has order 3.

Define $\theta_2(a, b) = (b, a)$ and $\vartheta(a, b, c) = (b + c - \min(a, c), \min(a, c), a + b - \min(a, c))$.

Let $\mathcal{R}_{\mathbf{i},\mathbf{j}} : \mathbb{Z}^N \to \mathbb{Z}^N$ be the maps indexed by $\mathbf{i}, \mathbf{j} \in \mathrm{Red}(w_0)$ satisfying $\mathcal{R}_{\mathbf{j},\mathbf{k}} \circ \mathcal{R}_{\mathbf{i},\mathbf{j}} = \mathcal{R}_{\mathbf{i},\mathbf{k}}$ such that if $\mathbf{i}$ and $\mathbf{j}$ are as in (B1) or (B2) then $\mathcal{R}_{\mathbf{i},\mathbf{j}}(a)$ is θ_2 or ϑ applied to the corresponding entries in $a = (a_1, \dots, a_N)$.

Define $\mathcal{L}$ as the set of tuples $v = (v_{\mathbf{i}})$ indexed by $\mathbf{i} \in \mathrm{Red}(w_0)$, where each $v_{\mathbf{i}} \in \mathbb{N}^N$ and $v_{\mathbf{j}} = \mathcal{R}_{\mathbf{i},\mathbf{j}}(v_{\mathbf{i}})$.

Given $v = (v_{\mathbf{i}}) \in \mathcal{L}$, define $\mathbf{wt}(v) = -\sum_{j=1}^N a_j \gamma_j$ where $v_{\mathbf{i}} = (a_1, a_2, \dots, a_N)$ for arbitrary $\mathbf{i}$ and

$$\gamma_1 = \alpha_{i_1}, \qquad \gamma_2 = s_{i_1}(\alpha_{i_2}), \qquad \gamma_3 = s_{i_1}s_{i_2}(\alpha_{i_3}), \qquad \cdots$$

Then $\{\gamma_1, \gamma_2, \dots, \gamma_N\} = \Phi^+$ and this formula does not depend on the choice of $\mathbf{i} \in \mathrm{Red}(w_0)$.

For each $i \in I$, choose a word $\mathbf{i} \in \mathrm{Red}(w_0)$ such that $i_1 = i$.

Then let $\varepsilon_i(v) = a_1$ where $v_{\mathbf{i}} = (a_1, \dots, a_N)$. We define $e_i(v) = 0$ if $a_1 = 0$.

If $a_1 \neq 0$ then $e_i(v) = v'$ where $v' \in \mathcal{L}$ is the unique element with $v'_{\mathbf{i}} = (a_1 - 1, a_2, \dots, a_N)$.

We define $\varphi_i(v)$ by requiring that $\varphi_i(v) - \varepsilon_i(v) = \langle \mathbf{wt}(v), \alpha_i^\vee\rangle$.

Finally, $f_i(v) = v'' \in \mathcal{L}$ where $v''_{\mathbf{i}} = (a_1 + 1, a_2, \dots, a_N)$.

Key fact: all of these definitions of $e_i(v)$, $f_i(v)$, $\varepsilon_i(v)$, $\varphi_i(v)$ are also independent of the choice of $\mathbf{i}$.

Theorem 1.1. In any simply-laced type, the crystal $\mathcal{L}$ is isomorphic to $\mathcal{B}_\infty$.

A *root partition* of a weight μ is a tuple $(k_\alpha)_{\alpha\in\Phi^+}$ in which $k_\alpha \in \mathbb{N}$ and $\sum_{\alpha\in\Phi^+} k_\alpha \alpha = \mu$.

The *Kostant partition function* $P(\mu)$ computes the number of root partitions of μ.

Corollary 1.2. One has $\mathrm{ch}(\mathcal{L}) = \mathrm{ch}(\mathcal{B}_\infty) = \sum_{\mu\in\Lambda^+} P(\mu) t^{-\mu} = \prod_{\alpha\in\Phi^+}(1 - t^{-\alpha})^{-1}$.

On seminormal crystals $\mathcal{B}$, there is a W-action in which s_i acts as the involution $\sigma_i : \mathcal{B} \to \mathcal{B}$ given by

$$\sigma_i(x) = \begin{cases} f_i^k(x) & \text{if } k \geq 0 \\ e_i^{-k}(x) & \text{if } k < 0 \end{cases} \qquad \text{where } k = \langle \mathbf{wt}(x), \alpha_i^\vee\rangle.$$

This formula can be zero on $\mathcal{L}$ so does not define a W-action. However, if we enlarge $\mathcal{L}$ to a crystal $\widehat{\mathcal{L}}$ whose elements $v = (v_{\mathbf{i}})$ allow views $v_{\mathbf{i}} \in \mathbb{Z}^n$ rather than just $v_i \in \mathbb{N}^n$, and we modify the definition of e_i accordingly to never act as zero, then we do get a Weyl group action in the expected way.

2 MV polytopes in type A_2

Our goal is to explain how the Lusztig parametrization of $\mathcal{B}_\infty$ encodes certain polytopes in the ambient vector space of the weight lattice. These will be call *MV polytopes* after Mirković and Vilonen.

A *polytope* is the higher dimension analogue of a polygon or polyhedron. The most accessible polytopes are the convex ones, which can be defined as intersections of a set of half-spaces in $\mathbb{R}^d$.

Here is the idea. Choose $v \in \mathcal{B}_\infty$ and let $w \in W$ have length r.

Select a reduced word $(i_1, i_2, \dots, i_r) \in \mathrm{Red}(w)$.

Then complete this to a reduced word $\mathbf{i} = (i_1, \dots, i_N) \in \mathrm{Red}(w_0)$ for the longest element of W.

Define positive roots $\gamma_k = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k})$ in the usual way.

The sequence $\gamma(\mathbf{i}) := (\gamma_1, \gamma_2, \dots, \gamma_N)$ is a permutation of Φ^+ and if $v_\mathbf{i} = (a_1, \dots, a_N)$ then we define

$$\mathbf{wt}(v, w) = \sum_{j=1}^{r} a_j \gamma_j \in \Lambda.$$

By the argument we saw last class, this definition is independent of $\mathbf{i}$.

We have $\mathbf{wt}(v, 1) = 0$ and $\mathbf{wt}(v, w_0) = -\mathbf{wt}(v)$.

We will show that these weights are the vertices of a convex polytope, to be denoted $\mathrm{MV}(v)$.

First, we verify this for type A_2. Assume we are working in the A_2 Cartan type. Write $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$.

Let $v_\mathbf{i} = (a, b, c)$ with $\mathbf{i} = (1, 2, 1)$ and $v_{\mathbf{i}'} = (a', b', c')$ with $\mathbf{i}' = (2, 1, 2)$.

These are the only reduced words for $w_0 \in W = S_3$ and we have

$$\gamma(\mathbf{i}) = (\alpha_1, \alpha_1 + \alpha_2, \alpha_2) \qquad \text{and} \qquad \gamma(\mathbf{i}') = (\alpha_2, \alpha_1 + \alpha_2, \alpha_1).$$

We compute $\mathbf{wt}(v, w)$ for the six elements of the Weyl group:

- If $w = 1$ then $\mathbf{wt}(v, w) = 0$.
- If $w = s_1$ then $\mathbf{wt}(v, w) = a\alpha_1$.
- If $w = s_1 s_2$ then $\mathbf{wt}(v, w) = (a + b)\alpha_1 + b\alpha_2$.
- If $w = s_2$ then $\mathbf{wt}(v, w) = a'\alpha_2$.
- If $w = s_2 s_1$ then $\mathbf{wt}(v, w) = (a' + b')\alpha_2 + b'\alpha_1$.
- If $w = s_1 s_2 s_1 = s_2 s_1 s_2 = w_0$ then $\mathbf{wt}(v, w) = (a + b)\alpha_1 + (b + c)\alpha_2 = (a' + b')\alpha_2 + (b' + c')\alpha_1$.

Last identity holds as we have $(a', b', c') = \vartheta(a, b, c)$.

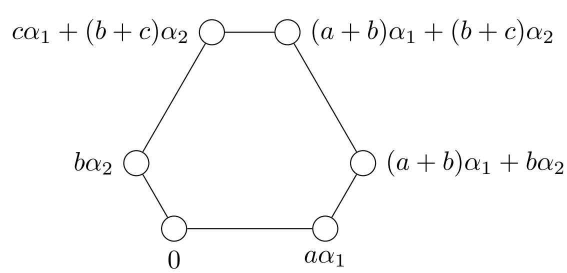
There are two cases. First assume $a \geq c$. Then

$$(a', b', c') = (b + c - \min(a, c), \min(a, c), a + b - \min(a, c)) = (b, c, a + b - c)$$

so we have:

- If $w = s_2$ then $\mathbf{wt}(v, w) = b\alpha_2$.
- If $w = s_2 s_1$ then $\mathbf{wt}(v, w) = (b + c)\alpha_2 + c\alpha_1$.

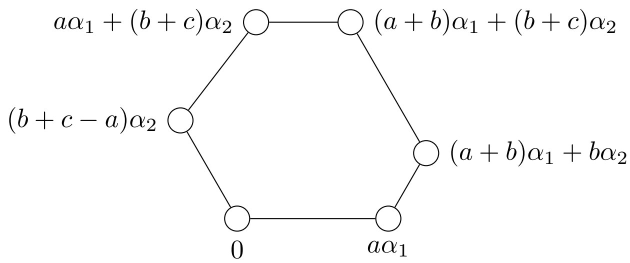
The corresponding picture is



In the other case we have $a \leq c$. Then $(a', b', c') = (b + c - a, a, b)$ so we have

- If $w = s_2$ then $\mathbf{wt}(v, w) = (b + c - a)\alpha_2$.
- If $w = s_2 s_1$ then $\mathbf{wt}(v, w) = (b + c)\alpha_2 + a\alpha_1$.

The corresponding picture is



For some a, b, c, these pictures can degenerate to situations where one or more of the six edges collapse to zero length. A polytope in the ambient vector space of the weight lattice that is a translate of one of these by a weight is called an *MV polytope* for Cartan type A_2. We will give the general definition later.

An important property of the MV polytope corresponding to $v \in \mathcal{B}_\infty$ is that it has a lowest weight $\lambda_{\text{low}} = 0$ and a highest weight λ_{high}. Given any reduced word $\mathbf{i} = (i_1, i_2, \dots, i_N) \in \text{Red}(w_0)$, let $v_{\mathbf{i}} = (a_1, \dots, a_N)$ be the corresponding view. Then the sequence of weights

$$\lambda_{\text{low}}, \qquad \lambda_{\text{low}} + a_1\gamma_1(\mathbf{i}), \qquad \lambda_{\text{low}} + a_1\gamma_1(\mathbf{i}) + a_2\gamma_2(\mathbf{i}), \qquad \dots$$

is a path from the lowest weight to the highest weight that travels around the edge of the polytope. The lengths of the segments of this path are $a_1, a_2, a_3, \dots$ so the polytope encodes in its dimensions every view in the Lusztig parametrization.

In the polytopes we drawn above for type A_2, the path for $\mathbf{i} = (1, 2, 1)$ proceeds around the right edge which the path for $\mathbf{i}' = (2, 1, 2)$ proceeds around the other side.

3 Tropical Plücker relations

So far we have assumed that (Φ, Λ) is a simply-laced Cartan type with ambient vector space V.

For simplicity, we now also assume that this Cartan type is semisimple, meaning the simple roots are a basis for V and the fundamental weights are the unique basis dual to the simple coroots.

Before we formalize MV polytopes we define a larger class of objects called *generalized Weyl polytopes*.

Write $\{\alpha_i : i \in I\}$ for the simple roots and $W = \langle s_i : i \in I \rangle$ for the Weyl group.

Let $\{\varpi_i^\vee : i \in I\}$ be the unique basis of V with $\langle \alpha_i, \varpi_j^\vee \rangle = \delta_{ij}$.

We call the elements $\varpi_i^\vee$ *fundamental coweights*.

A vector of the form $w(\varpi_i^\vee)$ for $w \in W$ is a *chamber coweight*.

Let $\mathsf{CW} = \{w(\varpi_i^\vee) : w \in W,\ i \in I\}$ be the set of chamber coweights.

Recall from HW3 that there exists a permutation of I, today written $i \mapsto i'$, such that $-w_0\alpha_i = \alpha_{i'}$.

Lemma 3.1. If $\nu^\vee$ is a chamber coweight then so is $-\nu^\vee$.

Proof. Suppose $\nu^\vee = w\varpi_i^\vee$. Then $\langle \alpha_j, w_0\varpi_{i'}^\vee \rangle = \langle w_0\alpha_j, \varpi_{i'}^\vee \rangle = \langle -\alpha_{j'}, \varpi_{i'}^\vee \rangle = -\delta_{i'j'} = -\delta_{ij}$.

Since Λ is semisimple, we have $w_0\varpi_{i'}^\vee = \varpi_i^\vee$ so $-\nu^\vee = -w\varpi_i^\vee = ww_0\varpi_{i'}^\vee \in \mathsf{CW}$. $\square$

Now let us fix a collection $M_\bullet$ of integers $M_{\nu^\vee} \in \mathbb{Z}$ for $\nu^\vee \in \mathsf{CW}$. Let

$$P(M_\bullet) = \{x \in \mathbb{R}\Lambda : \langle x, \nu^\vee \rangle \geq M_{\nu^\vee} \text{ for } \nu^\vee \in \mathsf{CW}\}.$$

For $w \in W$, there is a unique vector $\mu_w \in \mathbb{R}\Lambda$ such that

$$\langle \mu_w, w\varpi_i^\vee \rangle = M_{w\varpi_i^\vee} \text{ for all } i \in I,$$

since these equations force μ_w to be in the intersection of $|I|$ independent hyperplanes in a real vector space of dimension $d = |I|$, and this intersection is consequently a single point.

Later, we will show that $\mu_w = \mathbf{wt}(v, w)$ for some $v \in \mathcal{B}_\infty$.

From now on, $\boxed{\text{we assume that the } \mu_w \text{ are vertices of the polytope } P(M_\bullet)}$.

We do not require the vectors μ_w to be distinct, however.

Our assumption could fail if one of the $M_{\nu^\vee}$ is $\ll 0$ in which case the hyperplane defined by $M_{\nu^\vee} = \langle x, \nu^\vee \rangle$ could completely miss the polytope determined by the inequalities from the other chamber coweights. This would cause those μ_w on this hyperplane to not lie on the polytope.

Under our assumption, the set of vectors $\{\mu_w\}_{w \in W}$ is called a *GGMS datum*, named for Gelfand, Goresky, MacPherson, and Serganova, and $P(M_\bullet)$ is called a *generalized Weyl polytope*.

If λ is dominant, then the convex hull of the Weyl group orbit $W\lambda$ is a generalized Weyl polytope.

A dominant orbit $W\lambda$ is sometimes called a *Weyl polytope*.

Proposition 3.2. Assume the vectors μ_w are vertices of the polytope $P(M_\bullet)$. Then

$$M_{w\varpi_i^\vee} + M_{ws_i\varpi_i^\vee} \leq \sum_{j \neq i} -\langle \alpha_j, \alpha_i^\vee \rangle M_{w\varpi_j^\vee}. \tag{3.1}$$

We refer to (3.1) as the *edge inequalities*. It always holds that $-\langle \alpha_j, \alpha_i^\vee \rangle$ is a nonnegative integer.

Proof. We claim that $w\varpi_i^\vee + ws_i\varpi_i^\vee = \sum_{j\neq i} -\langle \alpha_j, \alpha_i^\vee \rangle w\varpi_j^\vee$.

To prove this, we may assume $w = 1$. Then, since Λ is semisimple, it suffices to show that both sides have the same inner product with alpha α_j. If $j = i$ then these inner products are both zero since

$$\langle \alpha_i, s_i\varpi_i^\vee \rangle = \langle s_i\alpha_i, \varpi_i^\vee \rangle = -\langle \alpha_i, \varpi_i^\vee \rangle = -1.$$

If $j \neq i$ then both inner products are $-\langle \alpha_j, \alpha_i^\vee \rangle$ since

$$\langle \alpha_j, s_i\varpi_i^\vee \rangle = \langle s_i\alpha_j, \varpi_i^\vee \rangle = \langle \alpha_j - \langle \alpha_j, \alpha_i^\vee \rangle \alpha_i, \varpi_i^\vee \rangle = -\langle \alpha_j, \alpha_i^\vee \rangle.$$

This proves our claim.

Now, we take the inner product of both sides or $w\varpi_i^\vee + ws_i\varpi_i^\vee = \sum_{j\neq i} -\langle \alpha_j, \alpha_i^\vee \rangle w\varpi_j^\vee$ with μ_w.

Since we have $\langle \mu_w, w\varpi_i^\vee \rangle = M_{w\varpi_i^\vee}$ and $\mu_w \in P(M_\bullet)$, it follows that $M_{ws_i\varpi_i^\vee} \leq \langle \mu_w, ws_i\varpi_i^\vee \rangle$ and

$$M_{w\varpi_i^\vee} + M_{ws_i\varpi_i^\vee} \leq \langle \mu_w, w\varpi_i^\vee \rangle + \langle \mu_w, ws_i\varpi_i^\vee \rangle = \sum_{j\neq i} -\langle \alpha_j, \alpha_i^\vee \rangle \langle \mu_w, ww_j^\vee \rangle = \sum_{j \neq i} -\langle \alpha_j, \alpha_i^\vee \rangle M_{w\varpi_j^\vee}$$

which is what we wanted to show. $\square$

An MV polytope will be a generalized Weyl polytope satisfying one additional family of relations, to be called the *tropical Plücker relations*. These are defined as the equalities

$$\boxed{M_{ws_i\varpi_i^\vee} + M_{ws_j\varpi_j^\vee} = \min\left(M_{w\varpi_i^\vee} + M_{ws_is_j\varpi_j^\vee}, M_{w\varpi_j^\vee} + M_{ws_js_i\varpi_i^\vee}\right)} \tag{3.2}$$

for all $w \in W$ and $i, j \in I$ with $\ell(ws_i) > \ell(w)$ and $\ell(ws_j) > \ell(w)$ and $\langle \alpha_i, \alpha_j^\vee \rangle = -1$.

(Recall that Φ is simply-laced so $\langle \alpha_i, \alpha_j^\vee \rangle \in \{0, -1\}$ for all $i \neq j$.)

Here $\ell : W \to \mathbb{N}$ is the length function of the Weyl group, which counts the length of any reduced word.

One has $\ell(ws_i) > \ell(w)$ if and only if $w(\alpha_i) \in \Phi^+$.

There are more complicated versions of the tropical Plücker relations for non-simply-laced types.

Lemma 3.3. If $P(M_\bullet)$ satisfies the tropical Plücker relations, then so does $P(M_\bullet) + \lambda$ for every $\lambda \in \Lambda$.

Proof. Fix $\lambda \in \Lambda$. Then $P(M_\bullet) + \lambda = P(M'_\bullet)$ where $M'_{\nu^\vee} = M_{\nu^\vee} + \langle \lambda, \nu^\vee \rangle$.

Assume $P(M_\bullet)$ satisfies the tropical Plücker relations.

To show that $P(M'_\bullet)$ also satisfies these relations, it is enough to check that

$$\langle \lambda, ws_i\varpi_i^\vee \rangle + \langle \lambda, ws_j\varpi_j^\vee \rangle = \langle \lambda, w\varpi_i^\vee \rangle + \langle \lambda, ws_is_j\varpi_j^\vee \rangle = \langle \lambda, ws_j\varpi_i^\vee \rangle + \langle \lambda, ws_js_i\varpi_i^\vee \rangle.$$

After replacing λ by $w\lambda$ we may assume $w = 1$. Then we actually have

$$s_i\varpi_i^\vee + s_j\varpi_j^\vee = \varpi_i^\vee + s_is_j\varpi_j^\vee = \varpi_j^\vee + s_js_i\varpi_i^\vee.$$

Note that the LHS is $s_i(x)$ for $x = \varpi_i^\vee + s_is_j\varpi_j^\vee$ and the RHS is $s_js_i(x)$. But

$$\langle \alpha_i, x \rangle = \langle \alpha_i, \varpi_i^\vee \rangle + \langle s_js_i\alpha_i, \varpi_j^\vee \rangle = 1 + \langle -\alpha_i - \alpha_j, \varpi_j^\vee \rangle = 1 - 1 = 0$$

and similarly $\langle \alpha_j, x \rangle = 0$. But this means that $s_i(x) = x - \langle x, \alpha_i^\vee \rangle \alpha_i = x$ and likewise $s_j(x) = x$. $\square$

Our next task is to verify the tropical Plücker relations for our MV polytopes of type A_2.

Thus, suppose (Φ, Λ) has type A_2. Assume s_1 and s_2 are right ascents of $w \in W$.

When $W = S_3$, this only occurs for $w = 1$ so the relation we need to check is

$$M_{s_1\varpi_1^\vee} + M_{s_2\varpi_2^\vee} = \min\left(M_{\varpi_1^\vee} + M_{s_1s_2\varpi_2^\vee}, M_{\varpi_2^\vee} + M_{s_2s_1\varpi_1^\vee}\right) \tag{3.3}$$

Since the weight lattice is required to be semisimple, our ambient vector space V is not $\mathbb{R}^3$ is this case but rather the quotient space $\mathbb{R}^3/\mathbb{R}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)$. The fundamental coweights are $\varpi_1^\vee = \mathbf{e}_1$ and $\varpi_2^\vee = \mathbf{e}_1+\mathbf{e}_2$.

Proposition 3.4. All translates of the MV polytopes in type A_2 satisfy the relation (3.3).

Proof. It suffices to show that the polytope $\mathrm{MV}(v)$ with vertices $\mathbf{wt}(v, w)$ for $v \in \mathcal{B}_\infty$ from page 2 satisfies (3.3). Suppose $v_{\mathbf{i}} = (a, b, c)$ for $\mathbf{i} = (1, 2, 1)$. Define $M_\bullet$ such that

- If $\nu^\vee = \varpi_1^\vee = \mathbf{e}_1$ then $M_{\nu^\vee} = 0$.
- If $\nu^\vee = \varpi_2^\vee = \mathbf{e}_1 + \mathbf{e}_2$ then $M_{\nu^\vee} = 0$.
- If $\nu^\vee = s_2\varpi_2^\vee = \mathbf{e}_1 + \mathbf{e}_3$ then $M_{\nu^\vee} = -b - c + \min(a, c)$.
- If $\nu^\vee = s_1\varpi_1^\vee = \mathbf{e}_2$ then $M_{\nu^\vee} = -a$.
- If $\nu^\vee = s_1s_2\varpi_2^\vee = \mathbf{e}_2 + \mathbf{e}_3$ then $M_{\nu^\vee} = -a - b$.
- If $\nu^\vee = s_2s_1\varpi_1^\vee = \mathbf{e}_3$ then $M_{\nu^\vee} = -b - c$.

Then $\mathrm{MV}(v) = P(M_\bullet)$ and it is straightforward to verify the necessary inequalities. $\square$

Definition 3.5. An *MV polytope* in $\mathbb{R}\Lambda$ is a generalized Weyl polytope $P(M_\bullet)$ that satisfies the tropical Plücker relations (3.2).

It follows from our proposition that any translate of the MV polytopes in type A_2 are indeed MV polytopes according to the general definition. Next time: more examples.

Why is (3.2) called the *tropical Plücker relation*?

For the $\mathrm{GL}(r,\mathbb{C})$ flag variety, the *Plücker coordinates* are in bijection with the chamber weights.

In detail, choose a matrix $g \in G = \mathrm{GL}(r,\mathbb{C})$ that may be projected onto the flag variety $X = G/B$. Let S be a proper nonempty subset of $I = \{1, 2, \dots, r\}$ and let p_S be the minor formed with g_{ij} for $i \in I$ and $r - |S| < j \leq r$. When $|S| = |S'|$ the ratio $p_S/p_{S'}$ is constant on the coset gB so these ratios are functions on X, and the Plücker coordinates p_S are homogeneous coordinates on X.

On the other hand S corresponds to a weight of the form $\sum_{i\in S} \mathbf{e}_i$ which are exactly the chamber weights.

Next week, we will describe a crystal structure on MV polytopes, which gives another model for $\mathcal{B}_\infty$.