1 Last time: MV polytopes

Assume (Φ, Λ) is a simply-laced and semisimple Cartan type, so that the simple roots all have the same length and are a basis for the ambient vector space V, and the fundamental weights are the unique basis dual to the simple coroots.

Write $\{\alpha_i : i \in I\}$ for the simple roots and $W = \langle s_i : i \in I\rangle$ for the Weyl group.

Let $\{\varpi_i^\vee : i \in I\}$ be the unique basis of V with $\langle \alpha_i, \varpi_j^\vee\rangle = \delta_{ij}$.

Let $\mathsf{CW} = \{w(\varpi_i^\vee) : w \in W,\ i \in I\}$ be the set of *chamber coweights*. We have $\mathsf{CW} = -\mathsf{CW}$.

Fix a collection $M_\bullet$ of integers $M_{\nu^\vee} \in \mathbb{Z}$ for $\nu^\vee \in \mathsf{CW}$. Let

$$P(M_\bullet) = \{x \in \mathbb{R}\Lambda : \langle x, \nu^\vee\rangle \geq M_{\nu^\vee} \text{ for } \nu^\vee \in \mathsf{CW}\}.$$

For $w \in W$, there is a unique vector $\mu_w \in \mathbb{R}\Lambda$ such that $\langle \mu_w, w\varpi_i^\vee\rangle = M_{w\varpi_i^\vee}$ for all $i \in I$.

Assume that the μ_w are vertices of the polytope $P(M_\bullet)$.

Then $P(M_\bullet)$ is called a *generalized Weyl polytope*. In this case, the following *edge inequalities* hold:

$$\boxed{M_{w\varpi_i^\vee} + M_{ws_i\varpi_i^\vee} \leq \sum_{j\neq i} -\langle \alpha_j, \alpha_i^\vee\rangle M_{w\varpi_j^\vee}.} \tag{1.1}$$

An *MV polytope* in $\mathbb{R}\Lambda$ is a generalized Weyl polytope satisfying the *tropical Plücker relations*:

$$\boxed{M_{ws_i\varpi_i^\vee} + M_{ws_j\varpi_j^\vee} = \min\left(M_{w\varpi_i^\vee} + M_{ws_is_j\varpi_j^\vee}, M_{w\varpi_j^\vee} + M_{ws_js_i\varpi_i^\vee}\right).} \tag{1.2}$$

Here, we require $w \in W$ and $i, j \in I$ with $\ell(ws_i) > \ell(w)$ and $\ell(ws_j) > \ell(w)$ and $\langle \alpha_i, \alpha_j^\vee\rangle = -1$.

Nontrivial fact: any translate $P(M_\bullet) + \lambda$ (for $\lambda \in \Lambda$) of an MV polytope $P(M_\bullet)$ is also an MV polytope.

Example 1.1. Assume (Φ, Λ) has type A_2. Write $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$.

Let $v \in \mathcal{B}_\infty$. Suppose $v_{\mathbf{i}} = (a, b, c)$ for $\mathbf{i} = (1, 2, 1)$ and $v_{\mathbf{i}'} = (a', b', c')$ for $\mathbf{i}' = (2, 1, 2)$.

These are the only reduced words for $w_0 \in W = S_3$ and we have

$$\gamma(\mathbf{i}) = (\alpha_1, \alpha_1 + \alpha_2, \alpha_2) \qquad \text{and} \qquad \gamma(\mathbf{i}') = (\alpha_2, \alpha_1 + \alpha_2, \alpha_1)$$

where $\gamma_k = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k})$. We define $\mathbf{wt}(v, w) = \sum_{j=1}^{\ell(w)} a_j\gamma_j \in \Lambda$.

These weights are the vertices of an MV polytope $\mathrm{MV}(v)$.

2 Crystal structure on MV polytopes

We will describe a crystal structure on MV polytopes of a given simply-laced, semisimple Cartan type.

Recall that $\mathcal{L}$ is the set of tuples $v = (v_{\mathbf{i}})$ with $v_{\mathbf{i}} \in \mathbb{N}^N$, indexed by reduced words $\mathbf{i} \in \mathrm{Red}(w_0)$.

There is a compatibility condition $v_{\mathbf{j}} = \mathcal{R}_{\mathbf{i},\mathbf{j}}(v_{\mathbf{i}})$ where $\mathcal{R}_{\mathbf{i},\mathbf{j}}$ are the transition maps defined in Lecture 21.

The weight of v with $v_{\mathbf{i}} = (a_1, a_2, \ldots, a_N)$ is $\mathbf{wt}(v) = -\sum_{j=1}^N a_j\gamma_j$ where $\gamma_j = s_{i_1}s_{i_2}\cdots s_{i_{j-1}}(\alpha_{i_j})$.

Let $\lambda_{\text{low}} \in \Lambda$ be a fixed element.

Given $w \in W$ with length $\ell(w) = l$, choose $\mathbf{i} = (i_1, \dots, i_N) \in \text{Red}(w_0)$ with $(i_1, \dots, i_l) \in \text{Red}(w)$.

Then define $\mathbf{wt}(v, w) = \lambda_{\text{low}} + \sum_{j=1}^{l} a_j \gamma_j$ where $\gamma_j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j})$.

This does not depend on the choice of reduced word $\mathbf{i}$.

We define $\text{MV}(v)$ to be the *convex hull* of the set of weights $\{\mathbf{wt}(v, w) \in \Lambda : w \in W\}$, that is,

$$\text{MV}(v) = \left\{ \sum_{w \in W} a_w \mathbf{wt}(v, w) : a_w \geq 0 \text{ and } \sum_{w \in W} a_w = 1 \right\}.$$

The *highest weight vector* of $\text{MV}(v)$ is $\lambda_{\text{high}} = \mathbf{wt}(v, w_0)$.

As the notation suggests, $\text{MV}(v)$ will be an MV polytope. But before showing this, our first task is to explain how $\text{MV}(v)$ is a generalized Weyl polytope. We will need the following lemma.

Write $\preceq$ for the partial order on Λ with $\mu \preceq \lambda$ if $\lambda - \mu = \sum_{i \in I} c_i \alpha_i$ with all $c_i \geq 0$.

Recall that W acts on the weight lattice by $s_i : \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$.

Lemma 2.1. Let $w, w' \in W$ and $v \in \mathcal{L}$. Then $w^{-1}\mathbf{wt}(v, w) \preceq w^{-1}\mathbf{wt}(v, w')$.

Proof. Suppose $i \in I$. We claim that $\mathbf{wt}(v, ws_i) - \mathbf{wt}(v, w) = c \cdot w(\alpha_i)$ for some $c \geq 0$.

We may assume $\ell(ws_i) > \ell(W)$ since otherwise we can interchange w and ws_i as $s_i(\alpha_i) = -\alpha_i$.

Pick a reduced word $(i_1, \dots, i_k)$ for w. Then $(i_1, \dots, i_k, i)$ is a reduced word for ws_i.

We complete this to a reduced word $\mathbf{i} \in \text{Red}(w_0)$. Suppose $v_{\mathbf{i}} = (a_1, \dots, a_N)$.

Then $\mathbf{wt}(v, ws_i) - \mathbf{wt}(v, w) = a_{k+1}\gamma_{k+1}$ where $\gamma_{k+1} = w(\alpha_i)$, so our claim holds with $c = a_{k+1} \geq 0$.

Turning to the lemma, let $w^{-1}w' = s_{j_1} \cdots s_{j_l}$ be a reduced expression.

Using the claim repeatedly, we deduce that

$$w^{-1}(\mathbf{wt}(v, w') - \mathbf{wt}(v, w)) = \sum_{j=1}^{l} w^{-1} \left(\mathbf{wt}(v, ws_{i_1} \cdots s_{i_j}) - \mathbf{wt}(v, ws_{i_1} \cdots s_{i_{j-1}}) \right)$$

is a nonnegative linear combination of the positive roots α_{i_1}, $s_{i_1}(\alpha_{i_2})$, $s_{i_1}s_{i_2}(\alpha_{i_3})$, ... ☐

Fix $v \in \mathcal{L}$. Let's explain how to realize $\text{MV}(v)$ as a set of the form

$$P(M_\bullet) = \{x \in \mathbb{R}\Lambda : \langle x, \nu^\vee \rangle \geq M_{\nu^\vee} \text{ for } \nu^\vee \in \mathsf{CW}\}$$

for some choice of integers $M_{\nu^\vee} \in \mathbb{Z}$ for $\nu^\vee \in \mathsf{CW}$.

Let $\nu^\vee = w\varpi_i^\vee$ be a chamber coweight. Then we have $\langle \mathbf{wt}(v, w'), \nu^\vee \rangle = \langle w^{-1}\mathbf{wt}(v, w'), \varpi_i^\vee \rangle$.

The inner product of $\varpi_i^\vee$ with any α_i is nonnegative, so $\langle \mathbf{wt}(v, w'), \nu^\vee \rangle \geq \langle \mathbf{wt}(v, w), \nu^\vee \rangle$ by the lemma.

This holds whenever $w\varpi_i^\vee = \nu^\vee$. Hence $\langle \mathbf{wt}(v, w'), \nu^\vee \rangle$ is minimized (over $w' \in W$) when w' belongs to

$$S_{\nu^\vee} = \{w' \in W : \nu^\vee = w'\varpi_i^\vee\},$$

which is a coset in W of the stabilizer $\{w' \in W : w'\varpi_i^\vee = \varpi_i^\vee\}$.

If $w', w'' \in S_{\nu^\vee}$ then

$$\langle \mathbf{wt}(v, w') - \mathbf{wt}(v, w''), \nu^\vee \rangle = \langle (w'')^{-1}\mathbf{wt}(v, w'), \varpi_i^\vee \rangle - \langle (w'')^{-1}\mathbf{wt}(v, w''), \varpi_i^\vee \rangle$$

is a nonnegative linear combination of simple roots by the lemma. But we also have

$$\langle \mathbf{wt}(v,w') - \mathbf{wt}(v,w''), \nu^\vee\rangle = \langle (w')^{-1}\mathbf{wt}(v,w'), \varpi_i^\vee\rangle - \langle (w')^{-1}\mathbf{wt}(v,w''), \varpi_i^\vee\rangle$$

which is a nonpositive linear combination of simple roots, by the same lemma.

Therefore $\langle \mathbf{wt}(v,w') - \mathbf{wt}(v,w''), \nu^\vee\rangle = 0$.

Thus if $w', w'' \in S_{\nu^\vee}$ then $\mathbf{wt}(v,w') - \mathbf{wt}(v,w'')$ is orthogonal to $\nu^\vee$. It follows that the convex hull $F_{\nu^\vee}$ of the set of weights $\mathbf{wt}(v,w')$ with $w' \in S_{\nu^\vee}$ is contained in a hyperplane $H_{\nu^\vee}$ orthogonal to $\nu^\vee$.

Define $M_{\nu^\vee}$ to be the constant value of $\langle \mathbf{wt}(v,w'), \nu^\vee\rangle$ for $w' \in S_{\nu^\vee}$.

Proposition 2.2. For each $v \in \mathcal{L}$, the set $\mathrm{MV}(v) = P(M_\bullet)$ is a generalized Weyl polytope.

Proof. Every $x \in \mathrm{MV}(v)$ is a convex combination of $\mathbf{wt}(v,w)$ for $w \in W$.

Thus, for each $\nu^\vee \in \mathsf{CW}$ we have $\langle x, \nu^\vee\rangle \geq M_{\nu^\vee}$ for $x \in \mathrm{MV}(v)$, with equality on the hyperplane $H_{\nu^\vee}$.

This means $F_{\nu^\vee}$ is a face of the convex hull of the weights $\mathbf{wt}(v,w)$ for $w \in W$, and $\mathrm{MV}(v) = P(M_\bullet)$. □

We follow this proposition with a stronger result.

Theorem 2.3. Let $v \in \mathcal{L}$. Then $\mathrm{MV}(v) = P(M_\bullet)$ is an MV polytope.

Moreover, for each $w \in W$ the unique element μ_w with $\langle \mu_w, w\varpi_i^\vee\rangle = M_{w\varpi_i^\vee}$ for all i is $\mu_w = \mathbf{wt}(v,w)$.

Proof. To show that $\mathrm{MV}(v)$ is an MV polytope we need to check the tropical Plücker relation.

We verified this in type A_2 last time.

Since (Φ, Λ) is simply-laced, it suffices to show how the general relation follows from the A_2 case.

For this, suppose $i, j \in I$ are such that $\langle \alpha_i, \alpha_j^\vee\rangle = -1$ and let $w \in W$ be such that $\ell(ws_i) > \ell(w)$ and $\ell(ws_j) > \ell(w)$. This implies (by standard but maybe not so obvious results in the theory of Coxeter systems) that $\ell(ws_is_js_i) = \ell(w) + 3$ and that $s_is_js_i = s_js_is_j$.

Choose a reduced word $(i_1, \dots, i_k)$ for w.

Then $(i_1, \dots, i_k, i, j, i)$ and $(i_1, \dots, i_k, j, i, j)$ are both reduced words for $ws_is_js_i$. Complete these to reduced words $\mathbf{i}$ and $\mathbf{j}$ for the longest element $w_0 \in W$. Write $v_\mathbf{i} = (a_1, \dots, a_N)$ and $v_\mathbf{j} = (b_1, \dots, b_N)$.

Define as usual $\vartheta(a,b,c) = (b + c - \min(a,c), \min(a,c), a + b - \min(a,c))$.

Then by the definition of the transition maps $\mathcal{R}_{\mathbf{i},\mathbf{j}}$ from Lecture 21, we have

$$(b_{k+1}, b_{k+2}, b_{k+3}) = \vartheta(a_{k+1}, a_{k+2}, a_{k+3})$$

while all other terms in $v_\mathbf{i}$ and $v_\mathbf{j}$ coincide. To simplify our notation let

$$(a,b,c) = (a_{k+1}, a_{k+2}, a_{k+3}) \qquad \text{and} \qquad (a',b',c') = (b_{k+1}, b_{k+2}, b_{k+3}) = \vartheta(a,b,c).$$

The simple roots α_i and α_j generate a type A_2 root system Φ' inside Φ, and the triple (a,b,c) is an element of the Lusztig parametrization of the $\mathcal{B}_\infty$ crystal for Φ'.

Using the discussion from last lecture, we know that the six (not necessarily distinct) weights

$$0, \qquad a\alpha_1, \qquad a\alpha_1 + b(\alpha_1+\alpha_2), \qquad a\alpha_1 + b(\alpha_1+\alpha_2) + c\alpha_2, \qquad a'\alpha_2, \qquad a'\alpha_2 + b'(\alpha_1+\alpha_2),$$

are the vertices of an MV polytope $P(M'_\bullet)$ for the type A_2 root system Φ', where $M'_{\nu^\vee}$ are the integers specified in the proof of Proposition 3.4 from Lecture 22.

The affine map $x \mapsto wx + \mathbf{wt}(v,w)$ takes these weights to the six weights $\mathbf{wt}(v,wy)$ for $y \in \langle s_i, s_j\rangle \cong S_3$.

Since translates of MV polytopes are still MV polytopes, it follows that we can deduce the tropical Plücker relation involving $M_{wy\varpi_i^\vee}$ and $M_{wy\varpi_j^\vee}$ for $y \in \langle s_i, s_j \rangle$ from the tropical Plücker relations for $P(M'_\bullet)$, which we already know to be valid. This proves that $\mathrm{MV}(v)$ is an MV polytope.

The argument that $\mu_w = \mathbf{wt}(v, w)$ is a little technical. The strategy is to first prove this for a particular MV polytope, given by the Weyl polytope whose vertices are the elements in the W-orbit of the Weyl vector $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$. This MV polytope is realized by taking $\lambda_{\mathrm{low}} = -\rho$ and $M_{\nu^\vee} = -1$ for all chamber coweights $\nu^\vee$. The element $v \in \mathcal{L}$ such that $\mathrm{MV}(v)$ is the Weyl polytope has $v_{\mathbf{i}} = (1, 1, \dots, 1)$ for all $\mathbf{i}$.

One can show that for the Weyl polytope one has $\mathbf{wt}(v, w) = \mu_w = w(-\rho)$. Specifically, choose a reduced word $(i_1, \dots, i_k)$ for $w \in W$ and complete it to a reduced word $\mathbf{i} \in \mathrm{Red}(w_0)$. Then

$$\mathbf{wt}(v, w) = \lambda_{\mathrm{low}} + \gamma_1(\mathbf{i}) + \dots + \gamma_k(\mathbf{i}) = -\rho + \sum_{j=1}^{k} \gamma_j(\mathbf{i}).$$

The roots $\gamma_1(\mathbf{i}), \dots, \gamma_k(\mathbf{i})$ are the elements of Φ^+ with $w^{-1}(\alpha) \in \Phi^-$ so

$$\mathbf{wt}(v, w) = \rho + \sum_{\substack{\alpha\in\Phi^+ \\ w^{-1}(\alpha)\in\Phi^-}} \alpha = w(-\rho).$$

On the other hand, this Weyl polytope is invariant under the action of W on Λ, and the action must be compatible with the action on chamber coweights. Therefore $\mu_w = w(\mu_1) = w(-\rho)$.

To deduce that $\mu_w = \mathbf{wt}(v, w)$ for a general MV polytope, one then argues that the desired identity remains valid, by continuity, as we deform the Weyl polytope through the space of data $M_\bullet$ that satisfy the edge and tropical Plücker relations. However, one needs to justify when any MV polytope can be obtained by this process of deformation. □

Let $\mathcal{B}$ be either $\mathcal{B}_\infty$ or $\mathcal{B}_\lambda$ where λ is a dominant weight. If $\mathcal{B} = \mathcal{B}_\infty$ then take $\lambda = 0$.

Then $\mathcal{B}$ is a highest weight crystal with highest weight λ, and we may embed $\mathcal{B} \hookrightarrow \mathcal{T}_\lambda \otimes \mathcal{L}$.

Suppose $v \in \mathcal{B}$ goes to $t_\lambda \otimes u \in \mathcal{T}_\lambda \otimes \mathcal{L}$ under this embedding. We define $v_{\mathbf{i}} = u_{\mathbf{i}} \in \mathbb{N}^N$ for $\mathbf{i} \in \mathrm{Red}(w_0)$.

The convex hull $\mathrm{MV}(v)$ of the weights $\mathbf{wt}(v, w) = \lambda_{\mathrm{low}} + \sum_{j=1}^{\ell(w)} a_j\gamma_j$ for $w \in W$ is an MV polytope.

For convenience set $\lambda_{\mathrm{low}} = \mathbf{wt}(v) := \lambda - \sum_{j=1}^{N} a_j\gamma_j$.

Then $\mathrm{MV}(v)$ has lowest weight $\mathbf{wt}(v)$ and highest weight $\lambda_{\mathrm{high}} = \lambda$.

The crystal structure on $\mathcal{B}$ induces a crystal structure on MV polytopes. Assume $f_i(v) \neq 0$.

To define the action of f_i on $\mathrm{MV}(v)$, choose a reduced word $\mathbf{i} \in \mathrm{Red}(w_0)$ with $i_1 = i$.

Then if $v_{\mathbf{i}} = (a_1, a_2, \dots, a_N)$, we have $(f_i(v))_{\mathbf{i}} = (a_1 + 1, a_2, \dots, a_N)$.

This means that the face of $\mathrm{MV}(v)$ adjacent to the lowest weight vector whose bounding hyperplane is

$$\langle x\varpi_i^\vee\rangle = M_{\varpi_i^\vee} \qquad \text{for } M_{\varpi_i^\vee} = \langle \mathbf{wt}(v), \varpi_i^\vee\rangle$$

is pushed out, increasing $M_{\varpi_i^\vee}$ by one. Likewise, the e_i operators act on the set of MV polytopes $\mathrm{MV}(v)$ by either pushing one bounding hyperplane in by one, or by sending $\mathrm{MV}(v) \mapsto 0$.

3 The $\star$-involution

Let G be a reductive Lie group with maximal torus T and maximal unipotent subgroup N^+. Then G has an involution $\star : G \to G$ such that $(g_1 g_2)^\star = g_2^\star g_1^\star$ that preserves T and N^+ and induces a bijection $\alpha \mapsto -w_0\alpha$ of the positive roots. For $G = \mathrm{GL}(n,\mathbb{C})$, the involution is $g^\star = w_0 g^T w_0$ where w_0 is the permutation matrix formed by reversed the order of the columns in the identity matrix.

The $\star$-involution on $\mathcal{B}_\infty$ is the tropicalization of the antiautomorphism of N^+ induced by $\star : G \to G$.

Let $\mathbf{i} = (i_1, \dots, i_N) \in \mathrm{Red}(w_0)$. Let $\mathbf{i}' = (i'_N, \dots, i'_1)$ where $i \mapsto i'$ is the permutation with $-w_0\alpha_i = \alpha_{i'}$.

Proposition 3.1. There is a weight-preserving bijection $\star : \mathcal{L} \to \mathcal{L}$ such that of $v_\mathbf{i} = (a_1, \dots, a_N)$ then

$$v^\star_{\mathbf{i}'} = (a_N, \dots, a_1).$$

Proof sketch. Check that if $v^\star_{\mathbf{i}'}$ is as given then $\mathcal{R}_{\mathbf{i}',\mathbf{j}'}(v^\star_{\mathbf{i}'}) = v^\star_{\mathbf{j}'}$ and $\mathbf{wt}(v^\star) = \mathbf{wt}(v)$. This is straightforward in the case when $\mathbf{i}'$ and $\mathbf{j}'$ are related by a single braid relation. The weight computation follows from the general identity $\gamma(\mathbf{i}') = (\gamma_N(\mathbf{i}), \dots, \gamma_1(\mathbf{i}))$. ☐

This means that the $\star$-involution of $\mathcal{L}$ is the tropicalization of the geometric map $\star : G \to G$.

In detail, the geometric map has the effect on N^+ of sending

$$x_{i_1}(a_1)\cdots x_{i_N}(a_n) \mapsto x_{i'_N}(a_N)\cdots x_{i'_1}(a_1)$$

and tropicalizing gives precisely the description in the previous result.

We have now defined the $\star$-involution for both $\mathcal{B}_\infty$ and $\mathcal{L}$, which are isomorphic crystals.

Theorem 3.2. If we identify $\mathcal{L} \cong \mathcal{B}_\infty$, then our two definitions of $\star$ coincide.

The proof of this result is omitted; it is due to Berenstein and Zelevinsky using the theory of quantum groups; Bump and Schilling outline a strategy for self-contained combinatorial proof of Chapter 15, using an embedding $\mathcal{L} \hookrightarrow \mathcal{B}_i \otimes \mathcal{L}$ analogous to $\psi_i : \mathcal{B}_\infty \hookrightarrow \mathcal{B}_i \otimes \mathcal{B}_\infty$.

As for $\mathcal{B}_\infty$ in Lectures 18-20, we may use $\star$ to define a modified crystal structure on the set $\mathcal{L}$.

We write $\varepsilon_i^\star(v) = \varepsilon_i(v^\star)$ and $\varphi_i^\star(v) = \varphi_i(v^\star)$ and define $e_i^\star$ and $f_i^\star$ by conjugating e_i and f_i by $\star$.

The $\star$-involution acts on MV polytopes in a simple way: $\mathrm{MV}(v^\star) = -\mathrm{MV}(v) := \{-x : x \in \mathrm{MV}(v)\}$.

Whereas f_i applied to $\mathrm{MV}(v)$ pushes out a bottom face, $f_i^\star$ pushes out a top face. In general,

$$f_i^\star : \mathrm{MV}(v) \mapsto -f_i(-\mathrm{MV}(v))$$

and likewise for $e_i^\star$. Finally,

$$\varepsilon_i(v) = \max\{k : \lambda_{\mathrm{low}} + k\alpha_i \in \mathrm{MV}(v)\} \qquad \text{and} \qquad \varepsilon_i^\star(v) = \max\{k : -k\alpha_i \in \mathrm{MV}(v)\}.$$

For proofs of these identities, see Proposition 15.30 in Bump and Schilling's book.

4 MV polytopes and the finite crystals $\mathcal{B}_\lambda$

Let $\lambda \in \Lambda^+$ be a dominant weight. Then we have a normal crystal $\mathcal{B}_\lambda$ with highest weight λ.

Since our Cartan type is simply-laced, $\mathcal{B}_\lambda$ is a Stembridge crystal.

The following results are due to Kamnitzer.

Proposition 4.1. Let P be an MV polytope whose highest weight is λ. Then (i) $M_{w_0 s_i \cdot \varpi_i^\vee} \geq \langle w_0\lambda, \varpi_i^\vee\rangle$ for all $i \in I$ if and only if (ii) P is contained in the Weyl polytope that is the convex hull of $W \cdot \lambda$.

The proof of this proposition appearing in the literature uses the affine Grassmannian and is not self-contained. Bump and Schilling note that a more direct proof would be desirable.

Theorem 4.2. Let λ be a dominant weight. Suppose that P is an MV polytope with highest weight λ. Then $P = \mathrm{MV}(v)$ for some $v \in \mathcal{B}_\lambda$ if and only if P is contained in the Weyl polytope that is the convex hull of the W-orbit of λ.

Proof. Some relatively straightforward algebraic manipulations show that

$$M_{w_0 s_{i'} \varpi_{i'}^\vee} = -\langle w_0\lambda, \varpi_{i'}^\vee\rangle = \langle \lambda, \alpha_i^\vee\rangle - \varepsilon_i^\star(v).$$

The criterion in the previous proposition for $\mathrm{MV}(v)$ to be contained in the convex hull of $W\cdot\lambda$ is equivalent to the assumption that $\varepsilon_i^\star(v) \leq \langle \lambda, \alpha_i^\vee\rangle$. This is equivalent to $v \in \mathcal{B}_\lambda$ by Theorem 3.4 in Lecture 19. □