1 Further topics

In this final lecture for our course, we survey some additional topics related to crystals and crystal bases.

This exposition follows Chapter 16 of Bump and Schilling's book; see the book for more references.

We provide only a brief overview of each topic, without attempt at comprehensiveness.

2 Kirillov-Reshetikhin crystals

In this course we have only discussed crystals associated to classical root systems.

These arise as the root systems of finite-dimensional Lie algebras.

There are all *affine root systems* associated to *affine Kac-Moody Lie algebras.*

For affine Lie algebras, there are finite-dimensional representations that are no longer highest weight.

There is a special class of finite-dimensional modules of the quantum groups associated to affine root systems, called *Kirillov-Reshetikin (KR) modules.* First studied around 1987.

Results of Okado and Schilling in 2007-2008 showed KR modules admit crystal bases.

These bases lead to notions of *Kirillov-Reshetikhin (KR) crystals* $B^{k,s}$.

KR crystals are labeled by positive integers k and s, where k is in the index set of the underlying classical root system. The characters of KR crystals satisfy a system of functional equations called the *Q-system.*

Here is the type A construction of KR crystals.

The index set of affine type $A_r^{(1)}$ is $I = \{0, 1, 2, \dots, r\}$.

By discarding the index 0, an $A_r^{(1)}$ crystal may be regarded as an A_r crystal.

Via this branching process, the KR crystal $B^{k,s}$ becomes the type A_r crystal $\mathrm{SSYT}_{r+1}(\mu)$ for $\mu = (s^k) = (s, s, s, \dots, s)$ the partition of sk with all parts equal to s.

The crystal operators f_0 and e_0 can be defined in terms of the *promotion operator* $\mathfrak{pr}$.

The *Bender-Knuth* involution BK_i on semistandard tableaux is defined as follows. For a tableau T, ignore all entries except i and $i+1$, and ignore all columns containing both i and $i+1$. You are then left with a disjoint set of rows of the form $i^a(i+1)^b$. Form $\mathsf{BK}_i(T)$ by changing each of these to be $i^b(i+1)^a$:

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 2 & 3 \\ \cline{1-4} 3 \\ \cline{1-1} \end{array} \quad \rightsquigarrow \quad \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & 2 & 3 & 3 \\ \hline & 2 & 2 & \\ \cline{1-4} \\ \cline{1-1} \end{array} \quad \rightsquigarrow \quad \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & 3 \\ \cline{1-4} 3 \\ \cline{1-1} \end{array} = \mathsf{BK}_2(T).$$

Then we define $\mathfrak{pr}(T) = \mathsf{BK}_r \circ \cdots \circ \mathsf{BK}_2 \circ \mathsf{BK}_1(T)$.

Each BK_i is an involution so $\mathfrak{pr}$ is invertible.

Finally, our crystal operators for $B^{k,s}$ are $f_0 = \mathfrak{pr}^{-1} \circ f_1 \circ \mathfrak{pr}$ and $e_0 = \mathfrak{pr}^{-1} \circ e_1 \circ \mathfrak{pr}$.

This formula can be seen as exploiting the rotational symmetry of the affine Dynkin diagram of $A_r^{(1)}$.

The $\mathfrak{pr}$ operator is complicated but this completely specifies the crystal $B^{k,s}$ for affine type A.

3 Littelmann path and alcove path models

The *Littelmann path model* is another way of constructing the elements of our usual Kashiwara crystals, in a way that comes with naturally associated crystal operators.

Here are the details. Fix a Cartan type (Φ, Λ).

Let $[0,1]_{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$. A *Littelmann path* is a piecewise linear mapping

$$\pi : [0,1]_{\mathbb{Q}} \to \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$$

such that $\pi(0) = 0$ and $\pi(1) \in \Lambda$. Two paths π_1, π_2 are considered to be the same if there is a piecewise-linear nondecreasing, surjective, continuous map $\phi : [0,1]_{\mathbb{Q}} \to [0,1]_{\mathbb{Q}}$ such that $\pi_1 = \pi_2 \circ \phi$.

There are associated *root operators* acting on these paths.

Let $h_\alpha(t) = \langle \pi(t), \alpha^\vee \rangle$ for each $\alpha \in \Phi$ and set $m_\alpha = \min_{t\in[0,1]_{\mathbb{Q}}}\{h_\alpha(t)\}$.

Let $\ell_\alpha(t)$ and $r_\alpha(t)$ be nondecreasing mappings on $[0,1]_{\mathbb{Q}}$ defined by

$$\ell_\alpha(t) = \min_{t\leq s\leq 1}\{1, h_\alpha(s) - m_\alpha\} \qquad\text{and}\qquad r_\alpha(t) = 1 - \min_{0\leq s\leq t}\{1, h_\alpha(s) - m_\alpha\}.$$

We have $\ell_\alpha(t) = 0$ until the last time that $h_\alpha(s) = m_\alpha$ and $r_\alpha(t) = 1$ after the first time that $h_\alpha(s) = m_\alpha$.

Define new paths π_ℓ and π_r by

$$\pi_\ell(t) = \pi(t) - \ell_\alpha(t)\alpha \qquad\text{and}\qquad \pi_r(t) = \pi(t) + r_\alpha(t)\alpha.$$

The *root operators* are now defined by

$$f_\alpha\pi = \begin{cases} \pi_\ell & \text{if } \ell(1) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad\text{and}\qquad e_\alpha\pi = \begin{cases} \pi_r & \text{if } r(0) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We can view Littelmann paths as a crystal for the operators $e_i = e_{\alpha_i}$ and $f_i = f_{\alpha_i}$.

Joseph and Kashiwara showed in the mid-1990s that this provides another construction for the Kashiwara crystals corresponding to highest weight representations of any (symmetrizable Kac-Moody) Lie algebra.

This is a useful perspective. For example, Stembridge originally formulated his crystal axioms as properties characterizing the Littelmann path model.

There is also an *alcove path model* due to Lenart and Postnikov from around 2008.

This is a discrete analogue of the path model, in which rather than allowing any piecewise-linear path from 0 to some weight λ, elements are given by sequences of adjacent alcoves cut out by affine hyperplanes.

4 Kyoto path model

Integrable representations for Kac-Moody Lie algebras are the "correct" generalization of finite-dimensional representations for finite-dimensional complex Lie algebras. The *Kyoto path model* is a construction of crystals corresponding to such representations, for affine Kac-Moody Lie algebras.

This is based on the notion of a *perfect crystal*.

The weight lattice for an affine Kac-Moody Lie algebra has the form

$$\Lambda = \mathbb{Z}\delta \oplus \bigoplus_{i\in I} \mathbb{Z}\varpi_i$$

where $I = \{0, 1, 2, \ldots, r\}$, ϖ_i is a fundamental weight, and δ is the *null root*.

The set of dominant weights is $\Lambda^+ = \bigoplus_{i\in I} \mathbb{N}\varpi_i$.

Each irreducible integrable representation of a Kac-Moody Lie algebra has a unique highest weight element whose weight is dominant. This also holds for the crystal derived from such a representation.

The set of *level ℓ weights* in Λ^+ is

$$\Lambda^+_\ell = \{\lambda \in \Lambda^+ : \langle c, \lambda\rangle = \ell\},$$

where c is the canonical central element of the affine Kac-Moody Lie algebra.

Write $\overline{\Lambda}$ for the weight lattice of the underlying classical Lie algebra, obtained by dropping the 0-node of the affine Dynkin diagram.

Suppose $\mathcal{B}$ is a crystal for our affine Kac-Moody Lie algebra. Write $\overline{\mathbf{wt}} : \mathcal{B} \to \overline{\Lambda}$ for the weight function of the crystal branched to the Cartan type obtained by dropping the 0-node of the affine Dynkin diagram.

Define $\varepsilon(b) = \sum_{i\in I} \varepsilon_i(b)\varpi_i$ and $\varphi(b) = \sum_{I\in I} \varphi_i(b)\varpi_i$.

Definition 4.1. For each positive integer ℓ, we say that $\mathcal{B}$ is *perfect* of level ℓ if

1. $\mathcal{B}\otimes\mathcal{B}$ is connected.
2. There exists $\overline{\lambda} \in \overline{\Lambda}$ such that
$$\overline{\mathbf{wt}}(\mathcal{B}) \subset \overline{\lambda} + \sum_{i\in I\setminus\{0\}} \mathbb{Z}_{\leq 0}\alpha_i$$
and there is a unique element $b \in \mathcal{B}$ with classical weight $\overline{\mathbf{wt}}(b) = \overline{\lambda}$.
3. For all $b \in \mathcal{B}$ it holds that $\langle c, \varepsilon(b)\rangle \geq \ell$.
4. For all $\omega \in \Lambda^+_\ell$, there exists unique elements $b_\omega, b^\omega \in \mathcal{B}$ with $\varepsilon(b_\omega) = \omega = \varphi(b^\omega)$.

The last condition is the most important. It implies that $\varepsilon, \varphi : \mathcal{B}_{\min} \to \Lambda^+_\ell$ are bijections where

$$\mathcal{B}_{\min} = \{b \in \mathcal{B} : \langle c, \varepsilon(b)\rangle = \ell\}.$$

The Kyoto path model constructs an integrable highest weight crystal $\mathcal{B}(\lambda)$ with highest weight element u_λ for each dominant weight $\lambda \in \Lambda^+$.

The model is recursive, making use of the following crystal isomorphism.

Given $\lambda_0 \in \Lambda^+_\ell$ and a perfect crystal $\mathcal{B}^0$ of level ℓ, there is a unique crystal isomorphism

$$\mathcal{B}(\lambda_0) \xrightarrow{\sim} \mathcal{B}^0 \otimes \mathcal{B}(\lambda_1)$$

with $u_{\lambda_0} \mapsto b^0 \otimes u_{\lambda_1}$, where b^0 is the unique element of $\mathcal{B}^0$ such that $\varphi(b^0) = \lambda_0$ and $\varepsilon(b^0) = \lambda_1$.

Iterating this isomorphism gives

$$\mathcal{B}(\lambda_0) \xrightarrow{\sim} \mathcal{B}^0 \otimes \mathcal{B}^1 \otimes \cdots \otimes \mathcal{B}^N \otimes \mathcal{B}(\lambda_{N+1}).$$

As $N \to \infty$, this models $\mathcal{B}(\lambda)$ as a semi-infinite tensor product of perfect crystals of level $\ell = \langle c, \lambda\rangle$.

Certain Kirillov-Reshetikhin crystals are perfect and can be used in this construction to build the infinite-dimensional highest weight crystals $\mathcal{B}(\lambda)$ for $\lambda \in \Lambda^+$.

5 Crystals on rigged configurations

Fix a (finite) Cartan type (Φ, Λ) of rank r.

A *rigged configuation* is a sequence $(\nu^{(1)}, \ldots, \nu^{(r)})$ of r partitions $\nu^{(i)}$ along with a set of *riggings*.

Riggings are nonnegative integer labels attached to each part of each partition, which are bounded by certain *vacancy numbers* depending on a related shape or tensor product of crystals.

This somewhat technical definition arises in the study of exactly solvable lattice models.

Rigged configurations index the solutions of certain *Bethe equations*.

A rigged configuration has a shape λ and a weight μ. In type A_r there is a shape- and weight-preserving bijection between rigged configurations and semistandard Young tableaux. Semistandard tableaux of shape λ and weight μ are also in bijection with the highest weight elements of weight λ in

$$\mathrm{SSYT}_{r+1}(\mu_1) \otimes \mathrm{SSYT}_{r+1}(\mu_2) \otimes \cdots$$

Based on this enumerative fact, it is natural to look for a generalization of the set of rigged configurations with a crystal structure, in which the rigged configurations defined above are the highest weight elements.

Schilling identified such crystals for simply-laced types around 2006. Schilling and Scimshaw have constructed analogous crystals of rigged configurations in the non-simply-laced cases.

There are also rigged configuration models for $\mathcal{B}_\infty$ and just a few years ago Salisbury and Scrimshaw described the $\star$-involution in terms of this model.

6 Modular branching rules of the symmetric group

The representation theory of the symmetric group S_n over $\mathbb{C}$ is well-understood.

All representations are completely reducible, the irreducible representations are indexed by partitions of n, and there are many explicit models for representations.

Many open questions remain about the representations of S_n over fields $\mathbb{F}$ of characteristic $\leq n$. For such fields, Maschke's theorem may fail and representations may not be completely reducible.

Modular representation theory refers to the study of representations in this setting.

Work of Dipper and James in the 1980s showed that the modular representation theory of S_n in prime characteristic p is closely related to the representation theory the corresponding *Iwahori-Hecke algebra* $\mathcal{H}_n(q)$ with its parameter $q = e^{2\pi i/p}$ specialized to a pth root of unity. Here, $\mathcal{H}_n(q)$ is a certain algebra over $\mathbb{C}(q)$ with a basis indexed by S_n, which becomes the group algebra for S_n when we set $q = 1$.

Assume $q = e^{2\pi i/p}$. Then the representation theory of $\mathcal{H}_n(q)$ is related in a surprising way to crystals, through an analogy where induction / restriction $\leftrightarrow$ to the operators e_i / f_i in affine $A^{(1)}_{p-1}$ crystals.

Lascoux, Leclerc, and Thibon conjectured in 1996 a very precise connection between crystal bases over affine Kac-Moody algebras and projective indecomposable modules over the Iwahori-Hecke algebras.

They also conjectured an efficient combinatorial algorithm for computing the multiplicities of the irreducible $\mathcal{H}_n(q)$-modules in the *Specht modules* $\mathbb{S}^\lambda$ — these multiplicities are called *decomposition numbers*.

Ariki proved these conjectures in 1996. This area has led to many interesting developments in representation theory in in the last two decades, such as the discovery of *KLR algebras*.

7 Crystals of Lie superalgebras

Lie superalgebras (or *graded Lie algebras*) are generalizations of Lie algebras that have a $\mathbb{Z}/2\mathbb{Z}$ grading, which reflects a geometry in which both commuting and anti-commuting variables can interact.

A *super vector space* $V = V_0 \oplus V_1$ is a vector space with a $\mathbb{Z}/2\mathbb{Z}$ grading.

The elements of V_0 or V_1 are *homogeneous*. The elements of V_0 are *even* while the elements of V_1 are *odd*.

If $a \in V$ is nonzero and homogeneous then let $|a| = i$ where $a \in V_i$.

One writes $\dim(V) = (m|n)$ where $m = \dim(V_0)$ and $n = \dim(V_1)$.

A *Lie superalgebra* is a supervector space with a bilinear operations $[\cdot,\cdot]$ that satisfies

$$[b, a] = -(-1)^{|a||b|}[a, b] \qquad \text{and} \qquad [a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]].$$

An example of $\mathfrak{gl}(m|n) = \mathrm{End}(V)$ for a super vector space $V = V_0 \oplus V_1$ with $\dim(V) = (m|n)$.

The $\mathbb{Z}/2\mathbb{Z}$ grading in this case is defined by setting

$$\mathrm{End}(V)_0 = \mathrm{End}(V_0) \oplus \mathrm{End}(V_1) \qquad \text{and} \qquad \mathrm{End}(V)_1 = \mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0).$$

The bracket operation is defined by $[a, b] = ab - (-1)^{|a||b|}ba$ for homogeneous elements $a, b \in \mathrm{End}(V)$, and extended to all elements by linearity.

Kac classified the finite-dimensional Lie superalgebras in the 1970s.

There are quantum groups associated to Lie superalgebras, leading to a theory of crystal bases.

Benkart, Kang, and Kashiwara gave the first tableau model of crystals for $\mathfrak{gl}(m|n)$ in 2000.

Jeong showed the existence of crystal bases for all finite-dimensional Lie superalgebras in 2001.

Like $\mathrm{GL}(n)$, the connected crystals for the Lie superalgebra $\mathfrak{gl}(n|m)$ have characters indexed by partitions, given by the *supersymmetric Schur functions* (sometimes called *hook Schur functions*).

The definition of these functions goes as follows.

Let λ be a partition and let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_m)$ be sequences of variables.

Denote the Schur function s_λ in all $n + m$ variables α_i and β_j as $s_\lambda(\alpha, \beta)$. Then

$$s_\lambda(\alpha, \beta) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu(\alpha) s_\nu(\beta).$$

The supersymmetric Schur functions $s_\lambda(\alpha|\beta)$ is similarly given by

$$s_\lambda(\alpha|\beta) := \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu(\alpha) s_{\nu^T}(\beta)$$

where ν^T is the transpose of ν. The terms $s_\mu(\alpha)$ and $s_{\nu^T}(\beta)$ are zero if $\ell(\mu) > n$ or $\ell(\nu^T) > m$.

Dualities analogous to Schur-Weyl duality exist for the crystals of $\mathfrak{gl}(m|n)$ and related Lie superalgebras.

There is also a generalization of the Cauchy identity for supersymmetric Schur functions:

$$\sum_\lambda s_\lambda(\alpha|\beta) s_\lambda(\gamma|\delta) = \frac{\prod_{i,j}(1 + \alpha_i\delta_j) \prod_{k,\ell}(1 + \beta_k\gamma_\ell)}{\prod_{p,q}(1 - \alpha_p\gamma_q) \prod_{s,t}(1 - \beta_s\delta_t)}.$$

This extends both the Cauchy identity and the dual Cauchy identity.

Another notable duality if *Sergeev duality* which is an analogue of Schur-Weyl duality.

It concerns the projective representations of the symmetric group, i.e., homomorphisms $S_n \to \mathrm{PGL}(\mathbb{C}^m)$.

These may be identified as ordinary representations of nontrivial central extensions of S_n.

Another notable Lie superalgebra is the *queer Lie superalgebra* $\mathfrak{q}(n)$.

Sergeev duality relates the representations of $\mathfrak{q}(n)$ to the projective representations of S_k in a way that is analogous to how Schur-Weyl duality relates the representations of $\mathrm{GL}(n)$ to the representations of S_k.

The theory of crystals bases for $\mathfrak{q}(n)$ has been developed only recently, by Grantcharov, Jung, Kang, Kashiwara, and Kim in the last decade. There are many interesting questions left to be explained concerning these crystals.

And that brings us to the end of the course! There are a couple of other further topics discussed in Chapter 16 in Bump and Schilling's book that I have not had time to include today (the *Nakajima monomial model*, *Tokuyama's formula*, etc.). Consult the book if you are interested.