**FINAL EXAMINATION SOLUTIONS – MATH 2121, FALL 2021.**

**Problem 1.** (30 points) This question has six parts.

(a) Find the general solution to the linear system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ x_2 + x_3 + x_4 = 3 \\ x_3 + x_5 = 2. \end{cases}$$

**Solution to part (a):**

The augmented matrix of this linear system is

$$A = \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{array}\right].$$

Its reduced echelon form is

$$\mathsf{RREF}(A) = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{array}\right].$$

This tells us that if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ is a solution then $x_4$ and $x_5$ are free variables while

$$x_1 = -3 - x_5, \quad x_2 = 1 - x_4 + x_5, \quad \text{and} \quad x_3 = 2 - x_5.$$

Therefore the general solution has the form

$$\boxed{x = \begin{bmatrix} -3-b \\ 1-a+b \\ 2-b \\ a \\ b \end{bmatrix} \quad \text{where } a, b \in \mathbb{R} \text{ are arbitrary.}}$$

(b) Find the standard matrix of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}$ with

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

**Solution to part (b):**

Since

$$T(x) = 2x_1 + 2x_3 + x_4 = \begin{bmatrix} 2 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

the standard matrix of $T$ is $\boxed{\begin{bmatrix} 2 & 0 & 2 & 1 \end{bmatrix}}$.

(c) Find the value of $h$ that makes the rank of the matrix

$$\begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 0 \\ 2 & 1 & h \\ 1 & 2 & 0 \end{bmatrix}$$

as small as possible.

**Solution to part (c):**

The second column is not a scalar multiple of the first, so the rank of the matrix is at least 2. The rank is exactly 2 if and only if the third column is a linear combination of the first two columns. There are real numbers $a, b \in \mathbb{R}$ such that

$$a\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2a \\ a+2b \\ 2a+b \\ a+2b \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ h \\ 0 \end{bmatrix}$$

if and only if

$$a = 1 \quad \text{and} \quad b = -a/2 = -1/2 \quad \text{and} \quad h = 2a + b = 2 - 1/2 = 3/2,$$

so the answer is $\boxed{h = 3/2}$.

(d) Find all $2 \times 3$ matrices $A$ that are in **reduced echelon form** and satisfy

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

**Solution to part (d):**

There was typo in this question in the printed exam, which referred to "$3 \times 2$ matrices" instead of $2 \times 3$ matrices. In its original form, the question wouldn't really make sense and a reasonable answer would be $\boxed{\text{no such } A \text{ matrices exist.}}$

However, this mistake was announced in the main lecture venue and we only saw a few cases of exams which did not take into account the updated wording of the question. With this correction, the answer is

$$\boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & a & -a-1 \\ 0 & 0 & 0 \end{bmatrix} \text{ for all } a \in \mathbb{R}, \quad \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.}$$

(e) Suppose $a, b, c, d, e \in \mathbb{R}$ are such that $ad - bc = 1$ and $e \neq 0$. Compute the inverse of

$$A = \begin{bmatrix} 0 & a & b \\ 0 & c & d \\ e & 0 & 0 \end{bmatrix}.$$

**Solution to part (e):**

One can check by multiplying the matrices that $\boxed{A^{-1} = \begin{bmatrix} 0 & 0 & 1/e \\ d & -b & 0 \\ -c & a & 0 \end{bmatrix}}$.

(f) Suppose $A$ is a $3 \times 3$ matrix with all real entries. The complex number $\lambda = 2 + 3i$ is an eigenvalue of $A$ and the trace of $A$ is $\mathrm{tr}(A) = 7$. What is the determinant of $A$?

**Solution to part (f):**

The trace is the sum of the eigenvalues and the determinant is the product of the eigenvalues (repeated with multiplicity). Since

$$\lambda_1 = 2 + 3i$$

is one eigenvalue it follows that

$$\lambda_2 = 2 - 3i$$

is another eigenvalue and so

$$\lambda_3 = \mathrm{tr}(A) - \lambda_1 - \lambda_2 = 7 - (2 + 3i) - (2 - 3i) = 7 - 4 = 3$$

is a third eigenvalue. Hence

$$\det(A) = \lambda_1 \lambda_2 \lambda_3 = (2 + 3i)(2 - 3i)3 = (4 - 9i^2)3 = (4 + 9)3 = 13 \times 3 = \boxed{39}.$$

**Problem 2.** (10 points) Do there exist two linearly independent vectors in $\mathbb{R}^4$ that are orthogonal to all three of the vectors

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ -5 \\ -2 \\ -7 \end{bmatrix}?$$

Find two such vectors if they exist, and otherwise explain why there are no such linearly independent vectors.

**Solution:**

The vectors in $\mathbb{R}^4$ orthogonal to the given vectors make up the null space of

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & -1 & 2 & 5 \\ 1 & -5 & -2 & -7 \end{bmatrix}.$$

The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the null space is 2-dimensional with a basis given by $\boxed{\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -8 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$.

It's a good idea to check that these vectors are in fact orthogonal to the three given vectors.

**Problem 3.** (10 points) This problem has two parts.

Suppose $A$ is a $3 \times 3$ matrix such that

$$A\begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}, \quad A\begin{bmatrix} 12 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad A\begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

(a) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

**Solution to part (a):**

The columns of $P$ should be linearly independent eigenvectors for $A$ and the diagonal entries of $D$ should be the corresponding eigenvalues. The given information provides exactly these eigenvectors and eigenvalues. There are multiple correct answers to this question but one would be

$$\boxed{P = \begin{bmatrix} 1 & 3 & 1 \\ -4 & 2 & -1 \\ 5 & 1 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}}.$$

(b) Determine if $\lim_{n\to\infty} A^n$ exists and compute its value if it does exist.

Explain how you found your answer to receive full credit.

**Solution to part (b):**

Since $A^n = PD^nP^{-1} = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4^n & 0 \\ 0 & 0 & 1/2^n \end{bmatrix} P^{-1}$ it follows that

$$\lim_{n\to\infty} A^n = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ -4 & 2 & -1 \\ 5 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ -4 & 2 & -1 \\ 5 & 1 & -1 \end{bmatrix}^{-1}.$$

This answer is not completely explicit but would be enough to get almost full credit on this question. To fully compute the answer we need to find the inverse of $P$. A shortcut for this is to observe that the columns of $P$ are orthogonal, so

$$P^\top P = \begin{bmatrix} 42 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Thus

$$P^{-1} = \begin{bmatrix} 42 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} P^\top = \begin{bmatrix} 1/42 & 0 & 0 \\ 0 & 1/14 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Without multiplying everything out, we can compute

$$\begin{aligned} P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} &= P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/42 & 0 & 0 \\ 0 & 1/14 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} P^\top \\ &= P\begin{bmatrix} 1/42 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^\top \\ &= \frac{1}{42} P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^\top \\ &= \frac{1}{42}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \\ &= \frac{1}{42}\begin{bmatrix} 1 & -4 & 5 \\ -4 & 16 & -20 \\ 5 & -20 & 25 \end{bmatrix} \end{aligned}$$

and so

$$\boxed{\lim_{n\to\infty} A^n = \frac{1}{42}\begin{bmatrix} 1 & -4 & 5 \\ -4 & 16 & -20 \\ 5 & -20 & 25 \end{bmatrix}}.$$

**Problem 4.** (20 points) This problem has four parts.

Suppose $A$ is a $3 \times 3$ matrix that has exactly two distinct (complex) eigenvalues given by $-1$ and $2$, and that has all three of the following vectors as eigenvectors:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

(a) Can the matrix $A$ be non-diagonalizable? If this is possible then give an example of such a matrix $A$, and otherwise explain why it is impossible.

**Solution to part (a):**

$\boxed{\text{No}}$. Any such matrix $A$ has 3 linearly independent eigenvectors so is diagonalizable.

(b) Can the matrix $A$ be non-invertible? If this is possible then give an example of such a matrix $A$, and otherwise explain why it is impossible.

**Solution to part (b):**

$\boxed{\text{No}}$. A matrix is non-invertible if and only if it has $0$ are an eigenvalue, but $A$ only has $-1$ and $2$ as eigenvalues.

(c) Can the matrix $A$ be orthogonal? (That is, can it hold that $A$ is invertible with $A^{-1} = A^\top$?) If this is possible then give an example of such a matrix $A$, and otherwise explain why it is impossible.

**Solution to part (c):**

$\boxed{\text{No}}$. Orthogonal matrices define length-preserving linear transformations, so can only have $\pm 1$ as eigenvalues. But $A$ has $2$ as an eigenvalue.

(d) Continue to suppose that $A$ is a $3 \times 3$ matrix that has exactly two distinct (complex) eigenvalues given by $-1$ and $2$, and that has all three of the following vectors as eigenvectors:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Can the matrix $A$ be symmetric? (That is, can it hold that $A = A^\top$?) If this is possible then give an example of such a matrix $A$, and otherwise explain why it is impossible.

**Solution to part (d):**

$\boxed{\text{Yes}}$. Suppose $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ are in the $-1$-eigenspace of $A$ while $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in the 2-eigenspace. Then

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$ (with eigenvalues $-1$, $-1$, and $2$ respectively), so the matrix

$$\boxed{A = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}}$$

is symmetric with the desired properties. This kind of answer is explicit enough to receive full credit on this problem.

**Problem 5.** (10 points) This question has two parts.

Consider the plane $P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 3x - y + 6z = 0 \right\}$ in $\mathbb{R}^3$.

(a) The subspace $P$ is 2-dimensional. Find an orthogonal basis for $P$.

**Solution to part (a):**

Two linearly independent vectors in $P$ are $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}$. The Gram-Schmidt process converts the second vector to

$$\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} - \frac{18}{10} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} - \begin{bmatrix} 9/5 \\ 27/5 \\ 0 \end{bmatrix} = \begin{bmatrix} -9/5 \\ 3/5 \\ 1 \end{bmatrix}.$$

After rescaling this vector, we get an orthogonal basis for $P$ given by

$$\boxed{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix}}.$$

(b) Find the vector in $P$ that is closest to $v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

**Solution to part (b):**

The desired vector is the projection of $v$ onto $P$ which is

$$\frac{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix}}{\begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix}} \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix}$$

which we can rewrite as

$$\frac{9}{10} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \frac{-16}{115} \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix} = \frac{1}{230} \left( 207 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} - 32 \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix} \right)$$

$$= \frac{1}{230} \begin{bmatrix} 495 \\ 525 \\ -160 \end{bmatrix}$$

$$= \boxed{\frac{1}{46} \begin{bmatrix} 99 \\ 105 \\ -32 \end{bmatrix}}.$$

**Problem 6.** (15 points) This question has three parts.

(a) Suppose $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Does the equation $Ax = b$ have an exact solution?

Find a solution or explain why none exists.

**Solution to part (a):**

The augmented matrix of the system $Ax = b$ is

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -1 & 1 \\ 2 & 2 & 2 \end{array}\right]$$

which has reduced echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Since the last column is a pivot column, the original linear system is inconsistent so has no exact solution.

(b) Again suppose $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Does the equation $Ax = b$ have a least-squares solution?

Find a solution or explain why none exists.

**Solution to part (b):**

The least-squares solutions to $Ax = b$ are the solutions to $A^\top Ax = A^\top b$. Since

$$A^\top A = \begin{bmatrix} 5 & 7 \\ 7 & 14 \end{bmatrix} \quad \text{and} \quad A^\top b = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

the augmented matrix of $A^\top Ax = A^\top b$ is

$$\left[\begin{array}{cc|c} 5 & 7 & 6 \\ 7 & 14 & 9 \end{array}\right]$$

which has reduced echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1/7 \end{array}\right].$$

Hence the unique least-squares solution to $Ax = b$ is $\boxed{x = \begin{bmatrix} 1 \\ 1/7 \end{bmatrix}}$.

(c) Suppose $A$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$.

Indicate which of the following are TRUE or FALSE.

You do not need to provide any justification for your answers.

**Correct answers will receive $1$ point, blank answers will recieve $0$ points, and incorrect answers will lose $1$ point.**

1. If $x \in \mathbb{R}^n$ has $A^\top Ax = A^\top b$ then it always holds that $Ax = b$.

   TRUE | **[FALSE]**

2. If $x \in \mathbb{R}^n$ has $Ax = b$ then it always holds that $A^\top Ax = A^\top b$.

   **[TRUE]** | FALSE

3. If the equation $Ax = b$ has no solution then $A^\top Ax = A^\top b$ might also have no solution.

   TRUE | **[FALSE]**

4. If the equation $Ax = b$ has a unique solution then $A^\top Ax = A^\top b$ also has a unique solution.

   **[TRUE]** | FALSE

5. If the equation $A^\top Ax = A^\top b$ has a unique solution $x$ then $Ax = b$ has at most one solution.

   **[TRUE]** | FALSE

**Problem 7.** (10 points)

Define $\mathbb{R}^{3\times 3}$ to be the set of all $3 \times 3$ matrices with all real entries.

The set $\mathbb{R}^{3\times 3}$ is a vector space. Let

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

and define $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ by the formula $T(A) = JAJ$. This is a linear function.

Find all real numbers $\lambda \in \mathbb{R}$ such that $T(A) = \lambda A$ for some $0 \neq A \in \mathbb{R}^{3\times 3}$. For each of these eigenvalues $\lambda$ find a basis for the subspace $\{A \in \mathbb{R}^{3\times 3} : T(A) = \lambda A\}$.

**Solution :**

First check that

$$T\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}.$$

Thus $T(T(A)) = A$ so if $T(A) = \lambda A$ then $\lambda^2 A = \lambda T(A) = T(T(A)) = A$. This means that the only possible eigenvalues of $T$ are $\lambda = 1$ or $\lambda = -1$. Both numbers are in fact eigenvalues. A basis for the 1-eigenspace is

$$\boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}.$$

A basis for the $-1$-eigenspace is

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}.$$

**Problem 8.** (15 points) This question has three parts.

(a) Compute the singular values of the matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

**Solution to part (a):**

The singular values of $A$ are the square roots of the eigenvalues of

$$A^\top A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}.$$

We have

$$\det(A^\top A - xI) = \det \begin{bmatrix} 9-x & -9 \\ -9 & 9-x \end{bmatrix} = (9-x)^2 - 9^2 = (-x)(18-x)$$

so the eigenvalues of $A^\top A$ are $\lambda = 0$ and $\lambda = 18$. Hence the singular values are

$$\boxed{\sigma_1 = \sqrt{18} > \sigma_2 = 0}.$$

(b) Suppose $A$ is a $2 \times 2$ matrix with a singular value decomposition

$$A = U\Sigma V^T$$

where $U$ and $V$ are orthogonal $2 \times 2$ matrices and

$$\Sigma = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

The first column of $U$ is the vector $\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$.

Draw a picture of the region in $\mathbb{R}^2$ given by

$$\left\{ Ax : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \text{ is a vector with } x_1^2 + x_2^2 \leq 1 \right\}.$$

Make your picture as detailed as possible to receive full credit.

**Solution to part (b):**

The correct answer is a picture of a solid ellipse centered at the origin with radius vectors $\pm \begin{bmatrix} -8 \\ 6 \end{bmatrix}$ and $\pm \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

(c) Find an orthonormal basis of $\mathbb{R}^3$ that contains the vector $\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.

**Solution to part (c):**

A basis for the orthogonal complement of the span of this vector is

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Performing the Gram-Schmidt process converts the second vector to

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{-4}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$$

so an orthogonal basis of our original orthogonal complement is

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}.$$

Converting these to unit vectors gives the desired orthonormal basis:

$$\boxed{\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}, \quad \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{45}}\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}}.$$