Summary

Quick summary of today's notes. Lecture starts on next page.

- If A and B are $n \times n$ matrices with $AB = I_n$ then $BA = I_n$ and $A^{-1} = B$.
- A *subspace* H of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ containing the zero vector that is closed under linear combinations. This means that $0 \in H$ and if $u, v \in H$ and $c \in \mathbb{R}$ then $u + v \in H$ and $cv \in H$.

  The *zero subspace* of $\mathbb{R}^n$ is the set $\{0\}$ containing just the zero vector $0 \in \mathbb{R}^n$.

- Let A be an $m \times n$ matrix.

  The *column space* of A is the span of the columns of A. Denoted $\mathrm{Col}\, A$. This is a subspace of $\mathbb{R}^m$.

$$\mathrm{Col}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbb{R}\text{-span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a \\ b \\ a \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\} \subseteq \mathbb{R}^4$$

  The *null space* of A is the set of vectors $\mathrm{Nul}\, A = \{v \in \mathbb{R}^n : Av = 0\}$. This is a subspace of $\mathbb{R}^n$.

$$\mathrm{Nul}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = y + 2z = 0 \right\} = \left\{ \begin{bmatrix} 0 \\ -2z \\ z \end{bmatrix} : z \in \mathbb{R} \right\} \subseteq \mathbb{R}^3.$$

- A *basis* for a subspace $H \subseteq \mathbb{R}^n$ is a linearly independent set of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ such that

$$H = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_p\}.$$

  The *standard basis* of $\mathbb{R}^n$ is $e_1, e_2, \dots, e_n$ where $e_i \in \mathbb{R}^n$ is the vector with 1 in row i and 0 in all other rows. Any subspace of $\mathbb{R}^n$ has a basis with at most n vectors.

- The pivot columns of an $m \times n$ matrix A form a basis for $\mathrm{Col}\, A$.
- Both A and $\mathsf{RREF}(A)$ have the same null space. Usually $\mathrm{Col}\, A \neq \mathrm{Col}\, \mathsf{RREF}(A)$.

  To find a basis for $\mathrm{Nul}\, A$, determine the indices $i_1, i_2, \dots, i_p$ of the non-pivot columns of A.

  Then there are unique vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ such that any

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad \text{with} \qquad \mathsf{RREF}(A)x = 0$$

  can be written as $x = x_{i_1} v_1 + x_{i_2} v_2 + \dots + x_{i_p} v_p$. The vectors $v_1, v_2, \dots, v_p$ are a basis for $\mathrm{Nul}\, A$.

  For example, if $\mathsf{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 4 & -1 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$ then any $x \in \mathbb{R}^5$ with $\mathsf{RREF}(A)x = 0$ has

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 + x_5 \\ x_2 \\ -2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

  The three vectors on the right are a basis for $\mathrm{Nul}\, A = \mathrm{Nul}\, \mathsf{RREF}(A)$.

1 Last time: inverses

The following all mean the same thing for a function $f : X \to Y$:

1. f is *invertible*.
2. f is one-to-one and onto.
3. For each $b \in Y$ there is exactly one $a \in X$ with $f(a) = b$.
4. There is a unique function $f^{-1} : Y \to X$, called the *inverse* of f, such that

$$f^{-1}(f(a)) = a \qquad \text{and} \qquad f(f^{-1}(b)) = b \qquad \text{for all } a \in X \text{ and } b \in Y.$$

Proposition. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and invertible then $m = n$ and T^{-1} is linear and invertible.

The following all mean the same thing for an $n \times n$ matrix A:

1. A is *invertible*.
2. A is the standard matrix of an invertible linear function $T : \mathbb{R}^n \to \mathbb{R}^n$.
3. There is a unique $n \times n$ matrix A^{-1}, called the *inverse* of A, such that

$$A^{-1}A = AA^{-1} = I_n \qquad \text{where we define } I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

4. For each $b \in \mathbb{R}^n$ the equation $Ax = b$ has a unique solution.
5. $\mathsf{RREF}(A) = I_n$
6. The columns of A are linearly independent and their span is $\mathbb{R}^n$.

Proposition. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix.

(1) If $ad - bc = 0$ then A is not invertible.

(2) If $ad - bc \neq 0$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Proposition. Let A and B be $n \times n$ matrices.

1. If A is invertible then $(A^{-1})^{-1} = A$.
2. If A and B are both invertible then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
3. If A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Process to compute A^{-1}

Let A be an $n \times n$ matrix. Consider the $n \times 2n$ matrix $\begin{bmatrix} A & I_n \end{bmatrix}$.

If A is invertible then $\mathsf{RREF}\left(\begin{bmatrix} A & I_n \end{bmatrix}\right) = \begin{bmatrix} I_n & A^{-1} \end{bmatrix}$.

So to compute A^{-1}, row reduce $\begin{bmatrix} A & I_n \end{bmatrix}$ to reduced echelon form, and then take the last n columns.

2 Stronger characterization of invertible matrices

Remember that a matrix can only be invertible if it has the same number of rows and columns.

Theorem. When A is an $n \times n$ matrix, the following are equivalent:

(a) A is invertible.

(b) The columns of A are linearly independent.

(c) The span of the columns of A is $\mathbb{R}^n$

Proof. We already know that (a) implies both (b) and (c).

Assume just (b) holds. Then A has a pivot position in every column, so $\mathsf{RREF}(A) = I_n$ since A has the same number of rows and columns. But this implies that A is invertible.

Similarly, if (c) holds then A has a pivot position in every row, so $\mathsf{RREF}(A) = I_n$ and A is invertible. □

Corollary. Suppose A and B are both $n \times n$ matrices. If $AB = I_n$ then $BA = I_n$.

This means that if we want to show that $B = A^{-1}$ then it is enough to just check that $AB = I_n$.

Proof. Assume $AB = I_n$. Then the columns of A span $\mathbb{R}^n$ since if $v \in \mathbb{R}^n$ then $Au = v$ for $u = Bv \in \mathbb{R}^n$, so A is invertible. Therefore $B = A^{-1}AB = A^{-1}I_n = A^{-1}$ so $BA = A^{-1}A = I_n$. □

Important note: this corollary only applies to *square matrices.*

3 Subspaces of $\mathbb{R}^n$

Let n be a positive integer. Write $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$.

Definition. Let H be a subset of $\mathbb{R}^n$. The subset H is a *subspace* if these three conditions hold:

1. $0 \in H$.
2. $u + v \in H$ for all $u, v \in H$.
3. $cv \in H$ for all $c \in \mathbb{R}$ and $v \in H$.

Common examples

$\mathbb{R}^n$ is a subspace of itself.

The set $\{0\}$ consisting of just the zero vector is a subspace of $\mathbb{R}^n$.

The empty set $\varnothing$ is *not* a subspace since it does not contain the zero vector.

A subset $H \subseteq \mathbb{R}^2$ is a subspace if and only if $H = \{0\}$ or $H = \mathbb{R}^2$ or $H = \mathbb{R}\text{-span}\{v\}$ for some $v \in \mathbb{R}^2$

The span of a set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Later, we will see that every subspace is the span of some set of vectors.

Example. The set

$$X = \left\{ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 1 \right\}$$

is *not* a subspace since $0 \notin X$.

Example. The set

$$H = \left\{ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0 \right\}$$

is a subspace since if $u, v \in H$ and $c \in \mathbb{R}$ then

$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0 + 0 = 0$$

and

$$cv_1 + cv_2 + cv_3 = c(v_1 + v_2 + v_3) = 0$$

so $u + v \in H$ and $cv \in H$.

Any matrix A gives rise to two subspaces, called the *column space* and *null space*.

Definition. The *column space* of an $m \times n$ matrix A is the subspace

$$\operatorname{Col} A \subseteq \mathbb{R}^m$$

given by the span of the columns of A.

Remark. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear function $T(x) = Ax$ then $\operatorname{Col} A = \mathsf{range}(T)$.

A vector $b \in \mathbb{R}^m$ belongs to $\operatorname{Col} A$ if and and only if $Ax = b$ has a solution.

Therefore $\operatorname{Col} A = \mathbb{R}^m$ if and only if $Ax = b$ has a solution for each $b \in \mathbb{R}^m$.

Definition. The *null space* of an $m \times n$ matrix A is the subspace

$$\operatorname{Nul} A \subseteq \mathbb{R}^n$$

given by the set of vectors $v \in \mathbb{R}^n$ with $Av = 0$.

Proof that $\operatorname{Nul} A$ *is a subspace.* If $u, v \in \operatorname{Nul} A$ and $c \in \mathbb{R}$ then $A(u + v) = Au + Av = 0 + 0 = 0$ and $A(cv) = c(Av) = 0$, so $u + v \in \operatorname{Nul} A$ and $cv \in \operatorname{Nul} A$. Thus $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$. □

Remark. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear function $T(x) = Ax$ then $\operatorname{Nul} A = \{x \in \mathbb{R}^n : T(x) = 0\}$.

The column space is a subspace of $\mathbb{R}^m$ **where** m **is the number of rows of** A.

The null space is a subspace of $\mathbb{R}^n$ **where** n **is the number of columns of** A.

A subspace can be completely determined by a finite amount of data. This data will be called a *basis*.

Definition. Let H be a subspace of $\mathbb{R}^n$. A *basis* for H is a set of vectors $\{v_1, v_2, \dots, v_k\} \subseteq H$ that are linearly independent and have span equal to H.

The empty set $\varnothing = \{\}$ is considered to be a basis for the zero subspace $\{0\}$.

Example. The set $\{e_1, e_2, \dots, e_n\} \subseteq \mathbb{R}^n$ where $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, and so on, is a basis for $\mathbb{R}^n$.

We call this the *standard basis* of $\mathbb{R}^n$.

Theorem. Every subspace H of $\mathbb{R}^n$ has a basis of size at most n.

Proof. If $H = \{0\}$ then $\varnothing$ is a basis.

Assume $H \neq \{0\}$. Let $\mathcal{B}$ be a set of linearly independent vectors in H that is as large as possible. The size of $\mathcal{B}$ must be at most n since any $n+1$ vectors in $\mathbb{R}^n$ are linearly dependent.

Let $w_1, w_2, \dots, w_k$ be the elements of $\mathcal{B}$. Since $\mathcal{B}$ is as large as possible, if $v \in H$ is any vector then $w_1, w_2, \dots, w_k, v$ are linearly dependent so we can write

$$c_1 w_1 + c_2 w_2 + \cdots + c_k w_k + cv = 0$$

for some numbers $c_1, c_2, \dots, c_k, c \in \mathbb{R}$ which are not all zero.

If $c = 0$ then this would imply that the vectors in $\mathcal{B}$ are linearly dependent. But the vectors in $\mathcal{B}$ are linearly independent, so we must have $c \neq 0$. Therefore

$$v = \tfrac{c_1}{c} w_1 + \tfrac{c_2}{c} w_2 + \cdots + \tfrac{c_k}{c} w_k.$$

This means that v is in the span of the vectors in $\mathcal{B}$. Since $v \in H$ is an arbitrary vector, we conclude that the span of the vectors in $\mathcal{B}$ is all of H, so $\mathcal{B}$ is a basis for H. □

Example. Let $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$.

How can we find a basis for $\operatorname{Nul} A$? Well, finding a basis for $\operatorname{Nul} A$ is more or less the same task as finding all solutions to the homogeneous equation $Ax = 0$. So let's first try to solve that equation.

If we row reduce the 3×6 matrix $\begin{bmatrix} A & 0 \end{bmatrix}$, we get

$$\begin{bmatrix} A & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(\begin{bmatrix} A & 0 \end{bmatrix}).$$

This tells us that $Ax = 0$ if and only if $\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0. \end{cases}$

Therefore $x \in \operatorname{Nul} A$ if and only if

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are a basis for $\operatorname{Nul} A$: we just computed that these vectors span the null space, and they are linearly independent since each has a nonzero entry in a row (namely, either row 2, 4, or 5) where the others have zeros. (Why does this imply linear independence?)

This example is important: the procedure just described works to construct a basis of $\operatorname{Nul} A$ for any matrix A. The size of this basis will always be equal to the number of free variables in the linear system $Ax = 0$. How to find a basis for $\operatorname{Nul} A$ is something you should remember at the end of this course.

Example. Let $B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

This matrix is in reduced echelon form. How to find a basis for $\operatorname{Col} B$?

The columns of B automatically span $\operatorname{Col} B$, but they might not be linearly independent.

The largest linearly independent subset of the columns of B will be a basis for $\operatorname{Col} B$, however.

In our example, the pivot columns 1, 2 and 5 are linearly independent since each has a row with a 1 where the others have 0s. These columns span columns 3 and 4, so a basis for $\operatorname{Col} B$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This example was special since the matrix B was already in reduced echelon form. To find a basis of the column space of an arbitrary matrix, we rely on the following observation:

Proposition. Let A be any matrix. The pivot columns of A form a basis for $\operatorname{Col} A$.

Proof sketch. Suppose A is $m \times n$. The reduced echelon form of A is obtained by multiplying A by an invertible matrix E on the left, so we can write $\mathsf{RREF}(A) = EA$.

If $a_1, a_2, \dots, a_k$ are the pivot columns of A, then $E \begin{bmatrix} a_1 & a_2 & \dots & a_k \end{bmatrix}$ is the $m \times k$ matrix $\begin{bmatrix} I_k \\ 0 \end{bmatrix}$ where the 0 means an $(m-k) \times k$ submatrix of zeros. These columns are linearly independent since if

$$\begin{bmatrix} a_1 & a_2 & \dots & a_k \end{bmatrix} v = 0$$

for $v \in \mathbb{R}^k$ then $0 = E \begin{bmatrix} a_1 & a_2 & \dots & a_k \end{bmatrix} v = \begin{bmatrix} I_k \\ 0 \end{bmatrix} v = \begin{bmatrix} v \\ 0 \end{bmatrix}$ which implies that $v = 0$.

Suppose $w \in \mathbb{R}^n$ is a non-pivot column of A. The definition of reduced echelon form implies that the corresponding column Ew of $\mathsf{RREF}(A) = EA$ is in the span of $Ea_1, Ea_2, \dots, Ea_k$. (Why?)

If $Ew = c_1 Ea_1 + \dots + c_k Ea_k$ then multiplying both sides by E^{-1} gives $w = c_1 a_1 + \dots + c_k a_k$ so w is in the span $a_1, a_2, \dots, a_k$. Therefore the span of the pivot columns of A is equal to $\operatorname{Col} A$.

Since the pivot columns are linearly independent and have span equal to $\operatorname{Col} A$, they form a basis. □

Example. The matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$$

is row equivalent to the matrix B in the previous example. Columns 1, 2, and 5 of A have pivots, so

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -9 \\ 2 \\ 1 \\ -8 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Col} A$.

Next time: we will show that if H is a subspace of $\mathbb{R}^n$ then all of its bases have the same size. The common size of each basis is the *dimension* of H.

4 Vocabulary

Keywords from today's lecture:

1. **Subspace** of $\mathbb{R}^n$

   A subset $H \subseteq \mathbb{R}^n$ such that $0 \in H$; if $u, v \in H$ then $u + v \in H$; and if $v \in H$, $c \in \mathbb{R}$ then $cv \in H$.

   Example: Pick any vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$. Then $\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_p\}$ is a subspace.

2. **Column space** of an $m \times n$ matrix A.

   The subspace $\operatorname{Col} A = \{Av : v \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$. The span of the columns of A.

   Example: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $\operatorname{Col} A = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \in \mathbb{R}^3 : x, y \in \mathbb{R} \right\}$.

3. **Null space** of an $m \times n$ matrix A.

   The subspace $\operatorname{Nul} A = \{v \in \mathbb{R}^n : Av = 0\} \subseteq \mathbb{R}^n$.

   Example: If $A = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & 0 \end{bmatrix}$ then $\operatorname{Nul} A = \left\{ \begin{bmatrix} 2x \\ x \\ y \end{bmatrix} \in \mathbb{R}^3 : x, y \in \mathbb{R} \right\} = \mathbb{R}\text{-span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

4. **Basis** of a subspace $H \subseteq \mathbb{R}^n$

   A set of linearly independent vectors in H whose span is H.

   Example: The vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are a basis for the subspace $\left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0 \right\}$.

   The **standard basis** of $\mathbb{R}^n$ consists of the vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.