Instructions: Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Wednesday, February 24**.

Throughout, all algebras and vector spaces are defined over an arbitrary field $\mathbf{k}$.

A *graded algebra* is an algebra with a subspace decomposition $A = \bigoplus_{n\geq 0} A[n]$ such that the unit element is in $A[0]$ and if $a \in A[m]$ and $b \in A[n]$ then $ab \in A[m+n]$.

The *Hibert series* of such an algebra is the generating function $h_A(t) = \sum_{n\geq 0} \dim A[n] \cdot t^n$. Often this converges to a rational function. For example, if $A = \mathbf{k}[x]$ and $A[n] = \mathbf{k}\text{-span}\{x^n\}$ then

$$h_A(t) = 1 + t + t^2 + \cdots = \tfrac{1}{1-t}.$$

1. Let Q be a finite quiver with vertex set $I = \{1, 2, \dots, n\}$. Let A_Q be the path algebra of Q. This is a graded algebra with $A_Q[n]$ given by the subspace spanned by all directed paths involving n edges. In particular, $A_Q[0] = \mathbf{k}\text{-span}\{p_i : i \in I\}$ where p_i is the empty path starting at vertex i.

   Find a closed formula for the Hilbert series of A_Q in terms of the adjacency matrix M_Q, defined as the $n \times n$ integer matrix whose entry in position (i, j) is the number of edges in Q from i to j.

2. Explain why a representation of a Lie algebra is the same thing as a representation of its universal enveloping algebra. In other words, for a representation (V, ρ) of a Lie algebra $\mathfrak{g}$, describe how to construct an associated representation of $U(\mathfrak{g})$. Then, from a representation of $U(\mathfrak{g})$, describe how to recover a Lie algebra representation of $\mathfrak{g}$. Finally, check that these ways of going between representations of $\mathfrak{g}$ and $U(\mathfrak{g})$ are inverse operations.

3. Let U, V, and W be vector spaces.

   Let π be the map $V \times W \to V \otimes W$ with $\pi(v, w) = v \otimes w$ for $v \in V$ and $w \in W$.

   (a) Check that π is bilinear. (b) Then show that if $f : V \times W \to U$ is any bilinear map, there exists a unique linear map $\tilde{f} : V \otimes W \to U$ with $f = \tilde{f} \circ \pi$. (c) Conclude that $f \mapsto \tilde{f}$ is a bijection from the set of bilinear maps $V \times W \to U$ to the set of linear maps $V \otimes W \to U$.

4. Let V and W be vector spaces.

   Suppose Z is a vector space and $\phi : V \times W \to Z$ is a bilinear map.

   Assume that whenever $g : V \times W \to U$ is a bilinear map to some vector space U, there exists a unique linear map $\tilde{g} : Z \to U$ with $g = \tilde{g} \circ \phi$.

   Show that there is a unique isomorphism $\theta : Z \xrightarrow{\sim} V \otimes W$ such that $\pi = \theta \circ \phi$.

5. Let V and W be vector spaces with bases $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$.

   Show that $\{v_i \otimes w_j\}_{(i,j)\in I\times J}$ is a basis for $V \otimes W$.

6. Show that $\alpha = \arccos(1/3)/\pi$ is not a rational number.

   The angles between incident faces in a regular tetrahedron are all $\pi\alpha$, while the angles between incident faces in a cube are all $\pi/2$. Using this, check that the Dehn invariant of a regular tetrahedron is nonzero while the Dehn invariant of a cube is always zero. (See Problem 1.51 in the textbook.)

7. Suppose $\mathfrak{g}$ is a Lie algebra with representations (V, ρ_V) and (W, ρ_W). Check that setting

   $$\rho_{V\otimes W}(x) = \rho_V(x) \otimes 1 + 1 \otimes \rho_W(x)$$

   for $x \in \mathfrak{g}$ makes $(V \otimes W, \rho_{V\otimes W})$ into a Lie algebra representation.

8. Classify the irreducible finite dimensional representations of the two dimensional Lie algebra $\mathfrak{g}$ with basis $\{X, Y\}$ and commutation relations $[X, X] = [Y, Y] = 0$ and $[X, Y] = Y = -[Y, X]$.

   Assume $\mathbf{k}$ is algebraically closed but consider the cases of zero and positive characteristic separately.

9. Let $\mathfrak{sl}(2)$ be the Lie algebra of 2×2 matrices over $\mathbb{C}$ with trace zero. (Here we assume $\mathbf{k} = \mathbb{C}$.)

   This Lie algebra has a basis given by $\left\{ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$.

   Show that for each integer $N > 0$, there exists a unique isomorphism class of N-dimensional irreducible representations of $\mathfrak{sl}(2)$. (Follow parts (a)-(f) of Problem 1.55 in the textbook.)