Instructions: Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Monday, March 8**.

Let A be an algebra defined over a field $\mathbf{k}$.

1. Show that if $W \subset V$ are finite-dimensional representations of A, then the characters of V, W, and V/W satisfy $\chi_V = \chi_W + \chi_{V/W}$.
2. Suppose $\mathbf{k} = \mathbb{R}$ and A is the algebra of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $f(x+1) = f(x)$ for all x. The product for this algebra is point-wise multiplication and the unit element is $f(x) = 1$.

   Let M be the A-module of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $f(x+1) = -f(x)$ for all x.

   Show that A and M are indecomposable, non-isomorphic A-modules.

   Show, however, that $A \oplus A \cong M \oplus M$ as A-modules.

   (Thus the Krull-Schmidt theorem fails for modules of infinite dimension.)
3. Show that if m and n are positive integers then $\mathrm{Mat}_m(\mathbf{k}) \otimes \mathrm{Mat}_n(\mathbf{k}) \cong \mathrm{Mat}_{mn}(\mathbf{k})$ as algebras.

In Exercises 4, 5, 6, and 7, assume $\mathbf{k} = \mathbb{C}$ and let V a finite-dimensional complex vector space with a symmetric bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{C}$. This form is said to be *nondegenerate* if for each $0 \neq v \in V$ there exists $w \in V$ with $(v, w) \neq 0$.

4. Show that the following are equivalent:
   (a) The form $(\cdot,\cdot)$ is nondegenerate;
   (b) For each $v \in V$ the map $v \mapsto (v,\cdot)$ is an isomorphism of vector spaces $V \to V^*$.
   (c) If $v_1, v_2, \ldots, v_n$ is a basis for V then the matrix $[(v_i, v_j)]_{1 \le i,j \le n}$ is invertible.

The *Clifford algebra* $\mathsf{Cliff}(V)$ is the quotient of the tensor algebra TV by the ideal $\langle v \otimes v - (v, v)1 : v \in V\rangle$.

5. Show that if $v_1, v_2, \ldots, v_n$ is a basis for V and $a_{ij} = (v_i, v_j)$ then $\mathsf{Cliff}(V)$ is isomorphic to the algebra generated by $v_1, v_2, \ldots, v_n$ subject to the relations $v_i v_j + v_j v_i = 2a_{ij}$ and $v_i^2 = a_{ii}$ for all $1 \le i, j \le n$ with $i \neq j$.
6. Suppose $(\cdot,\cdot)$ is nondegenerate. Show that $\mathsf{Cliff}(V)$ is semisimple. Show that if $\dim(V) = 2n$ is even, then $\mathsf{Cliff}(V)$ has exactly one isomorphism class of irreducible representations, and that if $\dim(V) = 2n + 1$, then $\mathsf{Cliff}(V)$ has exactly two isomorphism classes of irreducible representations. Show in both cases that all irreducible representations of $\mathsf{Cliff}(V)$ have dimension 2^n.
7. Show that $\mathsf{Cliff}(V)$ is not semisimple if $(\cdot,\cdot)$ is degenerate.

   What is $\mathsf{Cliff}(V)/\mathsf{Rad}(\mathsf{Cliff}(V))$ when $(\cdot,\cdot)$ is degenerate?

In Exercises 8, 9, 10, and 11, suppose (V, ρ_V) and (W, ρ_W) are two representations of A.

Here A is an algebra defined over an algebraically closed field $\mathbf{k}$.

Let U be the vector space $V \oplus W = \{(v, w) : v \in V,\ w \in W\}$

Suppose $f : A \to \mathrm{Hom}_{\mathbf{k}}(W, V)$ is a linear map. Define $\rho_f : A \to \mathrm{End}(U)$ to be the map with

$$\rho_f(a)(v, w) = (\rho_V(a)(v) + f(a)(w), \rho_W(a)(w)) \quad \text{for } a \in A,\ v \in V,\ w \in W.$$

8. Find a necessary and sufficient condition on $f(a)$ under which (U, ρ_f) is a representation of A.

   We denote the set of maps f satisfying this condition by $Z^1(W, V)$.

   This set is a vector space, and its elements are called (1-)cocycles. When $f \in Z^1(W, V)$, observe that (U, ρ_f) has a subrepresentation isomorphic to (V, ρ_V) and a quotient isomorphic to (W, ρ_W).

9. Let $F : W \to V$ be a linear map.

   Define the coboundary of F to be the function $dF : A \to \mathrm{Hom}_{\mathbf{k}}(W, V)$ with the formula

$$dF(a) = \rho_V(a) \circ F - F \circ \rho_W(a) \quad \text{for } a \in A.$$

   Show that $dF \in Z^1(W, V)$.

   Check that $dF = 0$ if and only if F is a morphism of representations $(W, \rho_W) \to (V, \rho_V)$.

   Let $B^1(W, V)$ be the subspace of coboundaries in $Z^1(W, V)$ and define $\mathsf{Ext}^1(W, V) = Z^1(W, V)/B^1(W, V)$.

10. Fix $f, g \in Z^1(W, V)$ and consider the A-representations (U, ρ_f) and (U, ρ_g).

    Show that if $f - g \in B^1(W, V)$ then (U, ρ_f) and (U, ρ_g) are isomorphic.

11. Continue the notation of the previous exercise.

    Further assume that V and W are finite dimensional and irreducible.

    Show that (U, ρ_f) and (U, ρ_g) are isomorphic if and only if f and g represent elements of $\mathsf{Ext}^1(W, V)$ that are scalar multiples of each other.

12. Assume $\mathbf{k} = \mathbb{C}$ and $A = \mathbb{C}[x_1, x_2, \ldots, x_n]$.

    Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ and $b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n$.

    Suppose V_a and V_b are 1-dimensional A-representations in which x_i acts as a_i and b_i, respectively.

    Find $\mathsf{Ext}^1(V_a, V_b)$ and classify the 2-dimensional representations of A.