Instructions: Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Monday, May 10**.

1. Let A be an algebra. Let $\mathscr{C}$ be the category of left A-modules.

   Show that the set of natural transformations from the identity functor on $\mathscr{C}$ to itself is an algebra isomorphic to the center $Z(A) := \{a \in A : ab = ba \text{ for all } b \in A\}$ of A.

2. Choose a field $\mathbf{k}$ and groups $H \subset G$. Let $\mathsf{Rep}(H)$ and $\mathsf{Rep}(G)$ be the categories of H- and G-representations defined over $\mathbf{k}$. For representations $V \in \mathsf{Rep}(H)$ and $W \in \mathsf{Rep}(G)$ we have notions of induced and restricted representations $\mathrm{Ind}_H^G(V) \in \mathsf{Rep}(G)$ and $\mathrm{Res}_H^G(W) \in \mathsf{Rep}(H)$.

   (a) Explain how $\mathrm{Ind}_H^G(\phi)$ and $\mathrm{Res}_H^G(\psi)$ should be defined for morphisms of representations ϕ and ψ so that $\mathrm{Ind}_H^G : \mathsf{Rep}(H) \to \mathsf{Rep}(G)$ and $\mathrm{Res}_H^G : \mathsf{Rep}(G) \to \mathsf{Rep}(H)$ become functors.

   (b) Show that Ind_H^G is left adjoint to Res_H^G.

   (c) Show that if G is finite then Ind_H^G is also right adjoint to Res_H^G.

3. Let A be a ring. Let Set be the category of sets (with arbitrary maps as morphisms) and let $\mathscr{C}$ be the category of left A-modules. Let $G : \mathscr{C} \to \mathsf{Set}$ be the forgetful functor. Define $F[X]$ for a set X to be the left A-module of maps $\phi : X \to A$ such that $\{x \in X : \phi(x) \neq 0\}$ is finite.

   (a) Explain how $F[\sigma]$ should be defined for maps $\sigma : X \to Y$ between sets so that F becomes a functor $\mathsf{Set} \to \mathscr{C}$. This is called a *free functor*.

   (b) Show that F is left adjoint to G.

4. Let A be an algebra defined over a field $\mathbf{k}$. Let $\mathscr{C}$ be the category of left A-modules. Let $\mathsf{Vect}_{\mathbf{k}}$ be the category of $\mathbf{k}$-vector spaces. Show that the forgetful functor $\mathscr{C} \to \mathsf{Vect}_{\mathbf{k}}$ is representable.

5. Let A be an algebra.

   A right A-module is *flat* if the function $M \otimes_A \bullet$ on the category of left A-modules is exact.

   (a) Show that any projective right A-module is flat.

   (b) Let $A = \mathbb{C}[x]$ and $M = \mathbb{C}[x, x^{-1}]$. Show that M is flat but not projective.

6. Let A be an algebra. Show that any left A-module M has a projective resolution.

7. Let A be an algebra.

   Show that $\mathsf{Tor}_0(M, N) \cong M \otimes_A N$ when M is a right A-module and N is a left A-module.

   Then show that $\mathsf{Ext}^0(M, N) \cong \mathrm{Hom}_A(M, N)$ when M and N are left A-modules.

8. Let $A = \mathbb{Z}$, $M = \mathbb{Z}/m\mathbb{Z}$, and $N = \mathbb{Z}/n\mathbb{Z}$. Compute $\mathsf{Tor}_i^A(M, N)$ and $\mathsf{Ext}_A^i(M, N)$ for all $i \geq 0$.

9. A *projective object* in an abelian category $\mathscr{C}$ is an object P such that the functor $\mathrm{Hom}_{\mathscr{C}}(P, \bullet)$ is exact. Show that the category of finite abelian groups has no nonzero projective objects.

10. Let A be a finitely generated commutative ring.

    Show that the abelian category $\mathscr{C}$ of finitely generated A-modules *has enough projectives* in the sense that every object is the quotient of a projective object.

    Deduce that every object in $\mathscr{C}$ has a projective resolution.