Math 5112 - Lecture #4

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Review: last time we saw "two categories of representations that are equivalent to subcategories of algebra repns"

⇝ motivates us to study repns of algebras (which we'll begin next time)

① Introduced notions of quivers $Q = (\underset{\text{vertices}}{I}, \overset{\text{edges}}{E})$ (directed graphs with self-loops and multiple edges allowed), quiver repns and the path algebra of Q

repn of a quiver $Q \equiv$ a choice of a vector space V_i for each vertex $i \in I$ and a linear map $\rho_{ij} : V_i \to V_j$ for each edge $i \to j$ in Q

path algebra of $Q \equiv$ vector space with basis the set of oriented paths in Q, multiplication given by concatenation

Key facts: if Q has finite set of vertices then path algebra has a unit and

$$\left\{\begin{array}{c}\text{quiver}\\ \text{repns of}\\ Q\end{array}\right\} \xleftrightarrow{1-1} \left\{\begin{array}{c}\text{algebra repns}\\ \text{of path algebra}\end{array}\right\}$$

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② Introduced notion of a Lie algebra

(a vector space $\mathfrak{g}$ with a bilinear form $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying $[x,x] = 0 \ \forall x \in \mathfrak{g}$ and

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0 \ \forall x,y,z)$$

Also discussed Lie algebra morphisms

(linear maps $\phi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\phi([x,y]) = [\phi(x), \phi(y)] \quad \forall\, x, y \in \mathfrak{g})$$

and Lie algebra representations

(pairs (V, ρ) where V is a vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a morphism of Lie algebras)

Here $\mathfrak{gl}(V)$ is the Lie algebra of linear maps $L : V \to V$ with bracket $[L_1, L_2] = L_1 \circ L_2 - L_2 \circ L_1$

Keys facts: (a) If A is any associative algebra then A becomes a Lie algebra for bracket $[a,b] = ab - ba$

$\mathfrak{gl}(V)$ is a special case of this (take $A = \mathrm{End}(V)$)

(b) "Lie algebras" are not "algebras"

For us, algebra $\equiv$ associative + unital

but the Lie bracket is neither

(c) for any Lie algebra $\mathfrak{g}$, there is a related algebra called the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ such that

Lie algebra repns of $\mathfrak{g}$ $\xleftrightarrow{1-1}$ algebra repns of $U(\mathfrak{g})$

Today: fundamentals of tensor products

much of algebra (or maybe all pure math) is subsumed by a precise understanding of tensor products

Basic setup: suppose V and W are k-vector spaces. (k can be any field for now).

The direct product of V and W as sets is

$$V \times W = \{ (a,b) \mid a \in V \text{ and } b \in W \}$$

This is just a set, no vector space structure.

Define $V * W$ to be the vector space with $V \times W$ as a basis.

Each element of $V * W$ is expressed uniquely as a finite K-linear combination of pairs $(a,b) \in V \times W$.

Def The ***tensor product*** $V \otimes W$ is the quotient of $V * W$ by the subspace spanned by all elements of the form

$$(v_1+v_2, w) - (v_1, w) - (v_2, w)$$

$$(v, w_1+w_2) - (v, w_1) - (v, w_2)$$

$$(av, w) - a(v,w)$$

$$(v, aw) - a(v,w)$$

for all $a \in K$, $v_1, v_2, v \in V$, $w_1, w_2, w \in W$

- $V \otimes W$ is a vector space
- For any $x \in V$ and $y \in W$ we write $x \otimes y$ for the image of (x, y) under the quotient map $V * W \to V \otimes W$
- Call $x \otimes y$ a <u>pure tensor</u>. Elements of $V \otimes W$ are sums of one or more pure tensors.
- Can have $x \otimes y = x' \otimes y'$ when $x \neq x'$, $y \neq y'$

Take $x' = -x$ and $y' = -y$

For any $a, b \in K$, we have $ax \otimes by = ab(x \otimes y)$

<u>HW Exercise:</u> If $\{v_i\}_{i\in I}$ is a basis for V and $\{w_j\}_{j\in J}$ is a basis for W then a basis for $V\otimes W$ is $\{v_i\otimes w_j\}_{(i,j)\in I\times J}$

<u>Another useful fact:</u> If $U, V,$ and W are all k-vector spaces, then there is a unique vector space isomorphism $U\otimes(V\otimes W)\to(U\otimes V)\otimes W$ that sends $a\otimes(b\otimes c)\mapsto(a\otimes b)\otimes c$ $\forall a\in U, b\in V, c\in W$

For this reason, we can consider n-fold tensor products $V_1\otimes V_2\otimes\cdots\otimes V_n$ if each V_i is a vector space, without worrying about the need for parentheses.

We also have a notion of the tensor product of linear maps. If $f: V \to V'$ and $g: W \to W'$ are linear maps then we define $f \otimes g$ to be the linear map $V \otimes W \to V' \otimes W'$ with

$$(f \otimes g)(a \otimes b) = f(a) \otimes g(b) \text{ for } a \in V,\ b \in W$$

To check that this is well defined:

$$(f \otimes g)\big((v_1+v_2)\otimes w - v_1 \otimes w - v_2 \otimes w\big)$$

$$= (f \otimes g)\big((v_1+v_2)\otimes w\big) - (f \otimes g)(v_1 \otimes w) - (f \otimes g)(v_2 \otimes w)$$

$$= \underbrace{f(v_1+v_2)\otimes g(w)} - f(v_1)\otimes g(w) - f(v_2)\otimes g(w) = 0 \checkmark$$

$$\forall\, v_1, v_2 \in V,\ w \in W$$

$$= (f(v_1) + f(v_2)) \otimes g(w)$$

$$= f(v_1)\otimes g(w) + f(v_2)\otimes g(w)$$

One checks similarly that we also get zero when applying $f \otimes g$ to

$$\begin{cases} v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2 & \forall v \in V, \; w_1, w_2 \in W \\ (av) \otimes w - a(v \otimes w) & \forall a \in K, \; v \in V, \; w \in W \\ v \otimes (aw) - a(v \otimes w) & \forall a \in K, \; v \in V, \; w \in W \end{cases}$$

Application: the tensor algebra of a vector space

Given a K-vector space V and an integer $n > 0$, let $V^{\otimes 0} \overset{\text{def}}{=} K$ and $V^{\otimes n} \overset{\text{def}}{=} \underbrace{V \otimes V \otimes V \otimes \cdots \otimes V}_{n \text{ factors}}$

Then set $\boxed{TV \overset{\text{def}}{=} \bigoplus_{n \geq 0} V^{\otimes n}}$ This is a vector space

It becomes an algebra with unit $1 \in K = V^{\otimes 0} \subset TV$ for the bilinear multiplication satisfying $ab \overset{\text{def}}{=} a \otimes b$ for $a \in V^{\otimes n}$, $b \in V^{\otimes m}$

Here, we identify

$$V^{\otimes n} \otimes V^{\otimes m} = (\underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ factors}}) \otimes (\underbrace{V \otimes V \otimes \cdots \otimes V}_{m \text{ factors}}) \xrightarrow{\sim} V^{\otimes(m+n)}$$

Call this structure TV the <u>tensor algebra</u> on V

<u>Fact</u> Suppose $\{v_1, v_2, \ldots, v_N\}$ is a basis for V.

Then there is a unique algebra isomorphism

$$TV \xrightarrow{\sim} K\langle x_1, x_2, \ldots, x_N \rangle$$ that sends each $v_i \mapsto x_i$

free algebra in non-commuting variables

(and more generally each tensor $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \mapsto x_{i_1} x_{i_2} \cdots x_{i_k}$)

This means that free algebras are "the same thing" as tensor algebras TV with a chosen basis for V

The tensor algebra TV has several notable quotients.

Def The quotient of TV by the two-sided ideal generated by the set $\{v\otimes w - w\otimes v \mid v,w\in V\}$ is called the **symmetric algebra SV**

Fact If $\{v_1, v_2, \ldots, v_N\}$ is a basis for V then there is a unique isomorphism $SV \to K[x_1, x_2, \ldots, x_N]$
polynomials in commuting variables
that sends each $v_i \mapsto x_i$

Polynomial algebras are "the same thing" as symmetric algebras SV with a chosen basis for V.

Def The quotient of TV by the two-sided ideal generated by the set $\{v \otimes v \mid v \in V\}$ is called the **exterior algebra** ΛV

($\$\backslash$wedge V$\$$)

In ΛV we have $(a+b) \otimes (a+b) = 0$

$$= \cancel{a \otimes a} + a \otimes b + b \otimes a + \cancel{b \otimes b}$$

$$= a \otimes b + b \otimes a \quad \text{for all } a \in V, b \in V$$

so $a \otimes b = -b \otimes a$

Thus we can think of elements of ΛV as sums of n-fold tensors where we can rearrange factors in each tensor by introducing sign changes.

Choosing a basis for V lets us identify

$\Lambda V \cong$ "polynomial" algebra in which the variables anti-commute, meaning $x_i x_j = -x_j x_i$

Finally:

<u>Def</u>. If V is a Lie algebra then its **universal enveloping algebra** is

$$U(V) = TV \Big/ \big\langle v \otimes w - w \otimes v - [v, w] \;\big|\; v, w \in V \big\rangle$$

this is an algebra

intersection of all two-sided ideals containing all of these elements

Tensor products of modules

Let A, B, C be k-algebras (k is any field)

Suppose $\boxed{V \text{ is a right } B\text{-module}}$ and

$\boxed{W \text{ is a left } B\text{-module}}$

Then we can form the tensor product

$$V \otimes_B W \overset{\text{def}}{=} (V \otimes W) \Big/ \left\langle vb \otimes w - v \otimes bw \;\middle|\; \begin{matrix} v \in V \\ w \in W \end{matrix} \;\; b \in B \right\rangle$$

"tensor product of V and W over B"

tensor product of vector spaces

subspace of $V \otimes W$ spanned by the given elements

Without more structure, $V \otimes_B W$ is just a vector space: not a module for B

already a right B-module

V is an (A,B)-bimodule if V is also a left A-module such that $(av)b = a(vb)$ $\forall \begin{matrix} a\in A \\ b\in B \\ v\in V \end{matrix}$

already a left B-module

W is a (B,C)-bimodule if W is also a right C-module such that $(bw)c = b(wc)$ $\forall \begin{matrix} b\in B \\ c\in C \\ w\in W \end{matrix}$

<u>Prop.</u> Assume V is an (A,B)-bimodule and W is a (B,C)-bimodule. Then $V\otimes_B W$ has a unique (A,C)-bimodule structure in which

$$\begin{cases} a(v\otimes_B w) \overset{\text{def}}{=} (av)\otimes_B w & \text{for } a\in A \quad v\in V \\ (v\otimes_B w)c \overset{\text{def}}{=} v\otimes_B (wc) & \qquad c\in C \quad w\in W \end{cases}$$

Here, we write $x \otimes_B y$ for the image of $x \otimes y \in V \otimes W$ under the quotient map $V \otimes W \to V \otimes_B W$

<u>Cor</u> If B is a commutative algebra, then

$$\left(\text{left } B\text{-modules}\right) \equiv \left(\text{right } B\text{-modules}\right)$$

$$\equiv \left((B,B)\text{-bimodules}\right)$$

So we can form the tensor product of two B-modules (by regarding ing these as (B,B)-bimodules) and the result will have a B-module structure

(*) If B is an algebra that is not commutative, then there is no natural definition of a tensor product for two left B-modules (or two right B-modules)