Math 5112 - Lecture #7

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Last time: k is an algebraically closed field

For integers $n > 0$, let $\mathrm{Mat}_n(k) = \left\{ \begin{array}{l} \text{algebra of} \\ n \times n \text{ matrices} \\ \text{over } k \end{array} \right\}$

Note if V is n-dim vector space over k then $\mathrm{End}(V) \cong \mathrm{Mat}_n(k)$

Suppose $A = \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(k)$ for some $d_1, d_2, \ldots d_r > 0$

Convenient to view $A \subset \mathrm{Mat}_n(k)$ for $n = d_1 + d_2 + \ldots + d_r$

Thm (A) For each index i, A has an irreducible representation $V_i \cong K^{d_i}$ (as vector spaces) and every finite-dimensional repn of A is a direct sum of copies of $V_1, V_2, \ldots, V_r$

Let V be a finite dimensional repn of some algebra A

Lemma There exists a finite filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$

where each V_i is a subrepn of V with V_i / V_{i-1} irreducible

Assume $\dim(A) < \infty$. The <u>radical</u> of A is

$\mathrm{Rad}(A) \overset{\text{def}}{=}$ set of elements in A that act as zero in every irreducible repn of A

$\overset{\text{(nontrivial)}}{=}$ largest nilpotent 2-sided ideal in A

<u>Thm (B)</u> A finite dimensional algebra A has finitely many irreducible representations $V_1, V_2, \dots, V_r$ up to isomorphism, each V_i has finite dimension, and

$$A / \mathrm{Rad}(A) \cong \bigoplus_{i=1}^{r} \mathrm{End}(V_i) \cong \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(k)$$

where $d_i = \dim(V_i)$

Each $\mathrm{End}(V_i)$ has dimension $d_i^2 = \dim(V_i)^2$ so

<u>Cor</u> If $\dim A < \infty$ then $\dim A - \dim \mathrm{Rad}(A) = \sum_{i=1}^{r} \dim(V_i)^2 \leq \dim A$

Examples

① Suppose $A = K[x]/(x^n)$ where $n \geq 1$.

$A = k\text{-span}\{1, x, x^2, x^3, \ldots, x^{n-1}\}$

$x^n = 0$ in $A \Rightarrow$ if (ρ, V) is a finite-dim repn of A then exists a basis for V in which matrix of $\rho(x)$ is strictly upper triangular

$\Rightarrow$ if V is irreducible then $\rho(x) = 0$ and $\dim V = 1$

Thus $A/\mathrm{Rad}(A) \cong \mathrm{End}(k) = k$ (can see directly that $\mathrm{Rad}(A) = (x)$)

② Suppose A is subalgebra of upper-triangular matrices in $\mathrm{Mat}_n(k)$

Let (V_i, ρ_i) be repn of A in which $V_i = k$ and $\rho_i(a) = a_{ii}$ (diagonal entry of a in row i)

for $i = 1, 2, \ldots, n$.

These repns are irreducible and pairwise-non-isomorphic.
They give all irred. repns of A (up to isomorphism) since $\mathrm{Rad}(A) = \left\{ \begin{array}{l} \text{strictly upper-triangular} \\ \text{matrices in } \mathrm{Mat}_n(k) \end{array} \right\}$ ← easy to see that this is the largest nilpotent 2-sided ideal in A

$\Rightarrow A/\mathrm{Rad}(A) \cong k^n \Rightarrow \exists\, n$ isomorphism classes of irred. A-repns.

Def. A finite-dimensional algebra A is called **semisimple** if $Rad(A) = 0$.

Recall that a repn of A is semisimple if it is a direct sum of irreducible subrepns.

Prop. Assume A is an algebra / k with $\dim A < \infty$.
The following are equivalent:

① A is semisimple

② $\sum_{i=1}^{r} \dim(V_i)^2 = \dim A$ where $V_1, V_2, \ldots, V_r$ are the distinct isomorphism classes of irreducible A-repns

③ $A \cong \bigoplus_{i=1}^{r} Mat_{d_i}(k)$ for some $d_1, d_2, \ldots, d_r > 0$

④ Any finite-dim repn of A is semisimple

⑤ The regular repn of A is semisimple

Pf. ① $\Leftrightarrow$ ② since $\dim A - \dim \mathrm{Rad}(A) = \sum_{i=1}^{r} \dim(V_i)^2$

① $\Leftrightarrow$ ③ by theorems (A) and (B).

Implication ① $\Rightarrow$ 3 is thm (B) exactly

③ + Thm (A) $\Rightarrow$ ② $\Rightarrow$ ①

Now we claim that $③ \underbrace{\Rightarrow}_{\text{by thm (A)}} ④ \underbrace{\Rightarrow}_{\text{trivial}} ⑤ \underbrace{\Rightarrow} ③$

Assume ⑤.

Can write $A = \bigoplus_{i=1}^{r} d_i V_i$ where $V_1, V_2, \dots, V_r$ are irreducible and pairwise non isomorphic

(this decomposition exists by ⑤)

Consider $\mathrm{End}_A(A) = \{\text{morphisms } A \to A \text{ as } A\text{-repns}\} = \mathrm{Hom}_A(A, A)$

Schur's lemma $\Rightarrow \begin{cases} \mathrm{End}_A(V_i) = k & \text{so } \mathrm{End}_A(d_i V_i) \cong \mathrm{Mat}_{d_i}(k) \\ \mathrm{Hom}_A(V_i, V_j) = 0 \text{ if } i \neq j & \text{so } \mathrm{Hom}_A(d_i V_i, d_j V_j) = 0 \text{ if } i \neq j \end{cases}$

$$\Rightarrow \mathrm{End}_A(A) = \mathrm{Hom}_A(A,A) = \bigoplus_{i,j} \underbrace{\mathrm{Hom}(d_iV_i, d_jV_j)}_{=0 \text{ if } i\neq j}$$

$$\cong \bigoplus_i \mathrm{Mat}_{d_i}(k)$$

Exercise: $(\mathrm{End}_A(A))^{op} \cong A$ (or $\mathrm{End}_A(A) \cong A^{op}$)

Last time: $\left(\bigoplus_i \mathrm{Mat}_{d_i}(k)\right)^{op} \cong \bigoplus_i \mathrm{Mat}_{d_i}(k)$ via transpose map

$\Rightarrow \underline{A \cong \bigoplus_i \mathrm{Mat}_{d_i}(k)}$ ← this is property 3 so 5 ⇒ 3. □

<u>Characters</u> Let A be an algebra.

Let (V,ρ) be a finite-dimensional repn of A.

The <u>character</u> of (V,ρ) is the linear map $\chi_{(V,\rho)}: A \to k$ with the formula $\chi_{(V,\rho)}(a) = \mathrm{trace}(\rho(a))$ for $a\in A$.

How to compute <u>trace</u> of $\phi \in End(V)$?

Choose a basis $e_1, e_2, \dots, e_n$ of V.

Then $trace(\phi) = \sum_{i=1}^{n} \left(\text{coefficient of } e_i \text{ in } \phi(e_i) \right)$

1. This defn does not depend on choice of basis
2. Always have $trace(\phi_1 \phi_2) = trace(\phi_2 \phi_1)$ for $\phi_1, \phi_2 \in End(V)$

   $\Rightarrow trace(\phi_1 \phi_2 \phi_1^{-1}) = trace(\phi_2)$ if ϕ_1 is invertible.
3. $\boxed{\text{If } (V_1, \rho_1) \cong (V_2, \rho_2) \text{ are finite. dim. } A\text{-repns then } \chi_{(V_1,\rho_1)} = \chi_{(V_2,\rho_2)}}$

If ρ is implicit, often write χ_V instead of $\chi_{(V,\rho)}$.

Let $[A,A] = k\text{-span}\{ [a,b] \overset{def}{=} ab - ba \mid a,b \in A \}$

<u>Fact</u> $[A,A] \subseteq kernel(\chi_{(V,\rho)})$

<u>Pf</u> Let $\chi = \chi_{(V,\rho)}$. Then $\chi(ab-ba) = trace(\rho(ab)) - trace(\rho(ba))$

$= trace(\rho(a)\rho(b)) - trace(\rho(b)\rho(a)) = 0$ □

Thm Characters of any list of non-isomorphic irred. fin.-dim. A-repns are linearly independent (and, in particular, are distinct).

(dim A not required to be finite)

Pf. Assume $(V_1, \rho_1), (V_2, \rho_2), \dots, (V_r, \rho_r)$ are pairwise non-isomorphic, irreducible, finite-dim. A-repns.

The map $\rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_r : A \to \mathrm{End}(V_1) \oplus \mathrm{End}(V_2) \oplus \cdots \oplus \mathrm{End}(V_r)$ is surjective. Thus if $\sum_{i=1}^{r} \lambda_i \chi_{(V_i, \rho_i)}(a) = 0$ for all $a \in A$

by density theorem

for some $\lambda_1, \lambda_2, \dots, \lambda_r \in K$, then $\sum_{i=1}^{r} \lambda_i \operatorname{trace}(M_i) = 0$ for any $M_i \in \mathrm{End}(V_i)$ chosen independently. Only possible if $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$

$\square$

Say that a character $\chi_{(V,\rho)}$ is irreducible if (V,ρ) is irreducible

Thm Assume A is semisimple and $\dim A < \infty$. Then the irreducible characters of A are a basis for $(A/[A,A])^*$

linear maps $A/[A,A] \to k$

Pf Each character χ has $[A,A] \subset \ker(\chi)$ so χ belongs to $(A/[A,A])^*$.

We have $A = \mathrm{Mat}_{d_1}(k) \oplus \cdots \oplus \mathrm{Mat}_{d_r}(k)$

$$\Rightarrow [A,A] = \bigoplus_{i=1}^{r} [\mathrm{Mat}_{d_i}(k), \mathrm{Mat}_{d_i}(k)]$$

Claim that $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] = \underbrace{\mathfrak{sl}_d(k)}_{\substack{d\times d \text{ matrices}/k \\ \text{with zero trace}}}$

Assuming the claim, we have

$A/[A,A] \cong k^r$ since $\mathrm{Mat}_d(k)/\mathfrak{sl}_d(k) \cong k$

But we know that A has r distinct irreducible characters (by Thm(A)) and these are linearly independent elements of $(A/[A,A])^*$ so must be a basis (as $\dim(A/[A,A])^* = \dim(A/[A,A]) = r$)

To prove claim: note that $E_{ij} = [E_{ik}, E_{kj}]$ for $i \neq j$

$E_{ii} - E_{i+1,i+1} = [E_{i,i+1}, E_{i+1,i}]$ where E_{ij} is elementary matrix with 1 in entry (i,j), 0 elsewhere. $\square$

Two general results

Assume $\dim A < \infty$

Jordan-Hölder thm: Let V be a finite-dim. repn of A. Suppose we have filtrations

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V \quad \text{and}$$

$$0 = V'_0 \subset V'_1 \subset V'_2 \subset \cdots \subset V'_m = V$$

where each V_i and V'_i is a subrepn with $W_i \overset{\text{def}}{=} V_i / V_{i-1}$ and $W'_i \overset{\text{def}}{=} V'_i / V'_{i-1}$ irreducible.

Then $n = m$ and $\exists$ permutation σ of $1, 2, 3, \ldots, n$ such that $W_{\sigma(i)} \cong W'_i$ for all i.

<u>Pf</u> (when char(k) $=0$)

Check that $\chi_V = \sum_{i=1}^{n} \chi_{W_i} \overset{(*)}{=} \sum_{i=1}^{n} \chi_{W_i'}$

by showing that $\chi_V = \chi_W + \chi_{V/W}$ for any subrepn W.

Deduce thm by linear independence of characters $\square$

(Proof doesn't work for char(k) $=p>0$ because multiplicities of irred. chars. could be multiples of p. Can handle this case by more involved inductive argument → see textbook.)

Call the common length $m=n$ of these filtrations the <u>length</u> of the repn V.

Assume $\dim A < \infty$

<u>Krull-Schmidt thm</u> If V is an A-repn with $\dim V < \infty$ then there is a decomposition of V, which is unique up to isomorphism and rearrangement of factors, as a direct sum of <u>indecomposable</u> A-repns

Proof next time. (Uniqueness claim is nontrivial part)