Math 5112 - Lecture #9

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Last time:

Jordan-Hölder theorem: any finite dim. repn V of an algebra has a filtration with irreducible quotients, and the length and quotients are uniquely det'd up to $\cong$ and permutation of indices

Krull-Schmidt theorem: any finite dim. repn V of an algebra A has (unique up to $\cong$ and rearrangement of factors) direct sum decomposition into indecomposable subrepns.

Comments: statements / proofs we gave for these theorems included assumption that the ambient algebra A has $\dim A < \infty$ and is defined over an alg. closed field k

① Both theorems still hold when $\dim A = \infty$

Pf The statements hold for V viewed as an A-repn iff they hold for V viewed as a repn of the fin. dim. alg. $\rho_V(A) \subset \mathrm{End}(V)$.

② Both thms still hold when k is not alg. closed.

Repns of tensor products:

If A and B are k-algebras then so is $A \otimes B$ in a natural way. (vector space tensor product)

If V is an A-repn and W is a B-repn then $V \otimes W$ is an $A \otimes B$-repn.

<u>Thm</u> If V, W are irreducible and finite-dimensional then so is $V \otimes W$ (as an $A \otimes B$-repn).

Up to $\cong$, all irreducible finite-dim. repns of $A \otimes B$ arise in this way.

Comments: previously, we reduced the proof of this to case when A and B are semisimple and finite dim $\rightsquigarrow$ hence (direct sums of) matrix algebras

In this concrete setting, theorem is easy to check directly.

For example $Mat_m(k)$ and $Mat_n(k)$ have unique irred repns k^m and k^n, and $k^m \otimes k^n \cong k^{mn}$ the unique irred repn of $Mat_{mn}(k) \cong Mat_m(k) \otimes Mat_n(k)$.

New topic: finite group representations

For a k-vector space V recall that $GL(V)$ is group of invertible linear maps $V \to V$.

Let G be a group. A (group) repn of G is a pair (V, ρ) where V is a vector space and $\rho : G \to GL(V)$ is a group homomorphism.

Key fact: we have seen that repns of G are the same thing as repns of the group algebra

$$k[G] = k\text{-span}\{a_g \mid g \in G\}$$

From now on, we will think of elements of $k[G]$ as formal (finite) linear combinations of group elements, writing $\sum_{g \in G} c_g g$ instead of $\sum_{g \in G} c_g a_g$

$c_g \in k$

"formal symbol indexed by $g \in G$"

We are interested in reps of finite groups G

In this case $k[G]$ has finite dimension.

First important question: when is $k[G]$ semisimple?

From now on, assume G is a finite group.

Maschke's thm Assume $\mathrm{char}(k)$ does not divide $|G| \overset{\text{def}}{=} \#$elements of G. Then $k[G]$ is semisimple.

Pf. Let (V, ρ) be a fin. dim. G-repn. (hence also a $k[G]$-repn). It suffices to check that V is a direct sum of irreducible subrepns. This clearly holds if (ρ, V) is irreducible so assume it isn't. Then V must have an irreducible subrepn W.

It is enough to show that V has another subrepn U such that $V = W \oplus U$ (by induction on dim)

Now it's easy to find a subspace $\tilde{U}$ that is <u>not</u> necessarily a subrepn but that has $V = W \oplus \tilde{U}$ as vector spaces.

Namely, choose a basis $w_1, w_2, \dots, w_m$ of W and extend to a basis $w_1, w_2, \dots, w_m, u_1, u_2, \dots, u_n$ for V and set $\tilde{U} = k\text{-span}\{u_1, u_2, \dots, u_n\}$

Key idea: let $\pi : V \to W$ be linear map with $\pi(w_i) = w_i$ and $\pi(u_i) = 0$. Then define

$$\sigma = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g^{-1})$$

Now take $U = \ker(\sigma)$. Claim that

① U is a subrepn and ② $V = W \oplus U$

① holds because $\sigma \rho(h) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g^{-1} h)$

set $x^{-1} = g^{-1} h$

$\Rightarrow g = hx$

$$= \frac{1}{|G|} \sum_{x \in G} \rho(hx) \circ \pi \circ \rho(x^{-1}) = \rho(h) \sigma$$

Thus $\sigma(u) = 0$ iff $\sigma \rho(h)(u) = 0$ for any $h \in G$
$u \in U$.

② Note that $\text{image}(\sigma) \subset W$ as W is subrepn.
Moreover, $\sigma(w) = w \ \forall w \in W$ since $\pi(w) = w \ \forall w \in W$.
$\Rightarrow \sigma^2 = \sigma$. Thus any $v \in V$ has

$$v = \underbrace{(v - \sigma(v))}_{\in U} + \underbrace{\sigma(v)}_{\in W}$$

and $W \cap U = 0$ since $x \in W \cap U$ has
$x = \sigma(x) = 0$. Thus $V = W \oplus U$ as needed.
$\square$

Cor Assume char(k) does not divide $|G|$.
Then there are finitely many isomorphism classes of irreducible G-repns $(V_1, \rho_1), (V_2, \rho_2), \dots, (V_r, \rho_r)$ and these all have finite dimension and

$$|G| = \sum_{i=1}^{r} (\dim V_i)^2.$$

Moreover $k[G] \xrightarrow{\cong} \bigoplus_{i=1}^{r} \mathrm{End}(V_i)$

via linear map with $\underset{\in G}{g} \longmapsto (\rho_1(g), \rho_2(g), \dots, \rho_r(g))$

Comment: Repn theory of finite-dim semisimple algebras is trivial in the sense that everything is just sum of matrix algebras.

What makes repn theory of finite groups interesting is the distinguished basis of $K[G]$ provided by G itself. Going from this basis to natural bases of $K[G]$ viewed as sum of matrix algebras is nontrivial.

Converse to Maschke's thm: If $K[G]$ is semisimple then $\mathrm{char}(K)$ does not divide $|G|$.

Pf. Assume $k[G]$ is semisimple.

The subspace $U \stackrel{\text{def}}{=} k\text{-span}\{\sum_{g \in G} g\}$ is a 1-dim'l subrepn of regular repn of $k[G]$.

semisimplicity $\Rightarrow$ there exists a complementary subrepn $V \subset k[G]$ with $k[G] = U \oplus V$

view k as trivial G-repn with $g \cdot c = c \ \forall g \in G, c \in k$

Define $\phi : k[G] \to k$ to be linear map that maps $V \to 0$ and send $\sum_{g \in G} g \mapsto 1_k$

Key insight: because U, V are subrepns, ϕ is a morphism of $k[G]$-repns. Thus $\underset{\substack{\uparrow \\ \text{in } k[G]}}{\phi(g)} = \phi(g \cdot 1) = g \cdot \underbrace{\phi(1)}_{\in k} = \phi(1)$

But this means that

$$1_k = \phi\left(\sum_{g\in G} g\right) = \sum_{g\in G} \phi(g) = \sum_{g\in G} \phi(1) = |G|\phi(1)$$

Thus $|G|$ must be invertible (and nonzero) in k, so $\mathrm{char}(k)$ must not divide $|G|$. $\square$

Characters of group repns

Assume G is a finite group. If (V,ρ) is a G-repn with $\dim V < \infty$ then its character is the map $\chi_{(V,\rho)}: G \to k$ with formula $\chi_{(V,\rho)}(g) = \operatorname{trace} \rho(g)$

Fact If $(V,\rho) \cong (V',\rho')$ then $\chi_{(V,\rho)} = \chi_{(V',\rho')}$

The conjugacy classes of G are the sets

$$X_g \overset{\text{def}}{=} \{ xgx^{-1} \mid x \in G \} \quad \text{for } g \in G$$

A class function of G is a map $G \to K$ that is constant on all conjugacy classes

$\Leftrightarrow$ f is class function iff $f(xgx^{-1}) = f(g)$ $\forall\, x, g \in G$

Fact Characters of G-reps are class functions

The character $\chi_{(V,\rho)}$ is irreducible if (V,ρ) is.

Prop. If $\text{char}(k)$ does not divide $|G|$ then the irreducible characters of G are a basis for the vector space of class functions.

Pf In this case $k[G]$ is semisimple so irred. characters are a basis for $(k[G]/[k[G],k[G]])^*$

but this is just $\left\{ \begin{array}{c} \text{linear maps} \\ f: G \to k \end{array} \middle| f(gh) = f(hg)\ \forall g,h \in G \right\}$

set $x = hg$
$\rightsquigarrow h = xg^{-1}$

$$= \left\{ \begin{array}{c} \text{linear maps} \\ f: G \to k \end{array} \middle| f(gxg^{-1}) = f(x)\ \forall x, g \in G \right\}$$

$= \text{class functions on } G$ $\square$

Cor. If $|G|$ is not divisible by char(k) then # isomorphism classes of irred. G-reprs is # distinct irred. characters of G and also is # distinct conjugacy classes of G.

Cor If char(k) $= 0$ then two G-repns (V, ρ) and (V', ρ') are isomorphic iff

$$\chi_{(V,\rho)} = \chi_{(V',\rho')}.$$

Def. G is <u>abelian</u> if $gh = hg$ for all $g, h \in G$

Fact If G is abelian then all irreducible G-repns are 1-dimensional

Pf In this case $k[G]$ is a commutative algebra (all irreps. of commutative algebras are 1-dim'l). $\square$

Suppose $f : V \to W$ is linear

means vector space of linear maps $V \to k$

Define $f^* : W^* \to V^*$ to be linear map with formula $f^*(\lambda) = \lambda \circ f$

If $f \in GL(V)$ then $f^* \in GL(V^*)$.

Suppose (V, ρ_V) is a G-repn.

Define $\rho_{V^*} : G \to GL(V^*)$ by formula

$$\rho_{V^*}(g) = \left(\rho_V(g)^*\right)^{-1} = \left(\rho_V(g)^{-1}\right)^* = \rho_V(g^{-1})^*$$

<u>Fact</u> If (V, ρ) is a repn then so is (V^*, ρ_{V^*})

Assume $\dim V < \infty$.

<u>Fact</u> $tr(f) = tr(f^*)$ so $\chi_{(V^*, \rho^*)}(g) = \chi_{(V, \rho)}(g^{-1})$ for all $g \in G$.

Fact $\chi_{(V,\rho)}(g)$ is sum of eigenvalues of $\rho(g)$, which must be roots of unity since $\rho(g)^{|G|} = \rho(g^{|G|})$
$= \rho(1_G) = 1$

Fact If $K = \mathbb{C}$ then

$$\chi_{(V,\rho)}(g^{-1}) = \overline{\chi_{(V,\rho)}(g)} = \chi_{(V^*,\rho^*)}(g)$$

for all $g \in G$. In this case $(V,\rho) \cong (V^*,\rho^*)$ if and only if $\chi_{(V,\rho)}(g) \in \mathbb{R}$ $\forall g$.

If (V, ρ_V) and (W, ρ_W) are G-reps then so is $(V \otimes W, \rho_{V\otimes W})$ where

$\rho_{V\otimes W}(g)$ is linear map $v \otimes w \longmapsto \rho_V(g)(v) \otimes \rho_W(g)(w)$

Fact If $\dim V < \infty$, $\dim W < \infty$ then

$$\chi_{(V\otimes W, \rho_{V\otimes W})} = \underbrace{\chi_{(V,\rho_V)} \chi_{(W,\rho_W)}}_{\text{pointwise multiplication of functions}}$$