Math 5112 - Lecture # 10

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Last time: representations and characters of (finite) groups.

Let k be an algebraically closed field.

A representation of a group G is an (algebra) repn (V, ρ) of the group algebra $k[G]$.

This means that $\rho(k[G]) \subseteq End(V)$ all linear maps $V \to V$

$\rho(G) \subseteq GL(V)$ invertible linear maps $V \to V$

Assume G is a finite group.

Maschke's theorem The group algebra $k[G]$ is $\boxed{\text{semisimple}}$ if and only if char(k) does not divide $|G|$.

(means all irreducible G-repns are finite-dim, and all finite-dim. G-repns are direct sums of irr. repns)

Assume (V,ρ) is a fin. dim. G-repn. Then its character is the linear map $\chi_{(V,\rho)}: k[G] \to k$ with $g \longmapsto \operatorname{trace}(\rho(g))$ for $g \in G$.

In this case $\dim V = \chi_{(V,\rho)}(1)$

sometimes called the degree

Say that $\chi_{(V,\rho)}$ is irreducible if (V,ρ) is.

Let $\mathrm{Irr}(G)$ denote set of irreducible characters of G. Some things that always hold:

1. If $(V,\rho) \cong (V',\rho')$ then $\chi_{(V,\rho)} = \chi_{(V',\rho')}$
2. Each $\chi = \chi_{(V,\rho)}$ is a <u>class function</u> on G, meaning a map $G \to K$ that is constant on conjugacy classes $\Leftrightarrow$ $\chi(ghg^{-1}) = \chi(h)$ for all $g,h \in G$

When $K[G]$ is semisimple, the following holds:

③ $\mathrm{Irr}(G)$ is a basis for vector space of class functions on G

④ If $\mathrm{char}(k)=0$, then $\chi_{(V,\rho)} = \chi_{(V',\rho')}$ if and only if $(V,\rho) \cong (V',\rho')$. Doesn't hold if $\mathrm{char}(k)>0$.

⑤ $$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$$

Ex If (V,ρ) is a G-repn with $\dim V = 1$, then $\chi_{(V,\rho)} = \rho$

Suppose $k=\mathbb{C}$ and G is cyclic group of order $n \geq 1$ generated by x

Let χ_m be map $\mathbb{C}[G] \to \mathbb{C}$ with $x^j \mapsto \zeta^{mj}$ where $\zeta = e^{\frac{2\pi\sqrt{-1}}{n}}$

Then $\mathrm{Irr}(G) = \{\chi_0, \chi_1, \chi_2, \dots, \chi_{n-1}\}$.

When a finite group is abelian (meaning that the group algebra is commutative), every irreducible repn is 1-dimensional. (this is true of all commutative algebras)

If V is a vector space then $V^* = \{\text{linear maps } V \to k\}$

If $f: V \to W$ is linear then $f^*: W^* \to V^*$ is map

$f^*: \lambda \mapsto \lambda \circ f$

If (V, ρ_V) is a repn of a group G then the <u>dual repn</u> is (V^*, ρ_{V^*}) where $\rho_{V^*}(g) = (\rho_V(g)^*)^{-1} = (\rho_V(g)^{-1})^* = \rho_V(g^{-1})^*$ for $g \in G$.

Fact $\chi_{(V^*, \rho_{V^*})}(g) = \chi_{(V,\rho_V)}(g^{-1}) \ \forall g \in G$

when $\dim V < \infty$. In this case we also have

$\overline{\chi_{(V,\rho)}(g)} = \chi_{(V,\rho)}(g^{-1}) \ \forall g \in G$ if $k = \mathbb{C}$.

If (V, ρ_V) and (W, ρ_W) are G-repns then the <u>tensor product repn</u> is $(V \otimes W, \rho_{V\otimes W})$ where

$\rho_{V\otimes W}(g)$ is the linear map $V \otimes W \to V \otimes W$ with

$v \otimes w \mapsto \rho_V(g)v \otimes \rho_W(g)w$ for $g \in G$, $v \in V$, $w \in W$.

Fact If $\dim V < \infty$, $\dim W < \infty$ then $\chi_{(V\otimes W, \rho_{V\otimes W})} = \chi_{(V,\rho_V)}\chi_{(W,\rho_W)}$

Remark A G-repn is a left $k[G]$-module.

$k[G]$ is often a noncommutative algebra.

Earlier we emphasized that if A is a noncommutative algebra then the tensor product of two left A-modules is not a well-defined left A-module in general.

How to explain the tensor product of group repns?

Solution: tensor product of two left A-modules does have structure of a left $A \otimes A$-module. In particular tensor product of (V, ρ_V) and (W, ρ_W) is a repn of $k[G] \otimes k[G]$. Special property of group algebras:

$k[G] \otimes k[G]$ has a subalgebra $k\text{-span}\{g \otimes g \mid g \in G\} \cong k[G]$

Hence any $k[G] \otimes k[G]$-repn can be viewed as a $k[G]$-repn $\rightsquigarrow$ this is how we define $(V, \rho_V) \otimes (W, \rho_W)$

<u>Today</u>: more special properties of characters.

Everywhere today, G is a finite group and $k = \mathbb{C}$.

Given any functions $f_1, f_2 : G \to \mathbb{C}$, define

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

This form $(-,-)$ is Hermitian and positive definite

linear in f_1
conjugate linear in f_2

means (f, f) is positive and real for $f \neq 0$

<u>Thm</u> $\mathrm{Irr}(G)$ is an orthonormal basis for class functions on G under $(\cdot,\cdot)$
irreducible chars of G

Specifically, $(\chi,\psi) = \begin{cases} 1 & \text{if } \chi=\psi \\ 0 & \text{if } \chi\neq\psi \end{cases}$ for $\chi,\psi\in \mathrm{Irr}(G)$

Moreover, for any fin. dim. G-repns (V,ρ_V), (W,ρ_W), we have $(\chi_{(V,\rho_V)},\chi_{(W,\rho_W)}) = \dim \underbrace{\mathrm{Hom}_G(W,V)}$

morphisms of $k[G]$-repns $W\to V$

<u>Pf</u> Suffices to prove this by Schur's lemma.

Let $\chi_V = \chi_{(V,\rho_V)}$ and $\chi_W = \chi_{(W,\rho_W)}$.

Then $(\chi_V,\chi_W) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\underbrace{\chi_W(g^{-1})}_{=\overline{\chi_W(g)}} = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\underbrace{\chi_{W^*}(g)}_{=\chi_W(g^{-1})} = \chi_{V\otimes W^*}(\Pi)$

where we define $\pi \overset{\text{def}}{=} \frac{1}{|G|} \sum_{g \in G} g \in k[G]$.

Note that $g\pi = \pi$ for all $g \in G$.

If X is any G-repn and $x \in X$ then

$$\pi x \in X^G \overset{\text{def}}{=} \{ v \in X \mid gv = v \ \forall g \in G \}.$$

is a subrepn of X

If X is irreducible then X^G is either X or 0

Thus π acts on any G-repn X as projection $X \twoheadrightarrow X^G$.

$\Rightarrow \chi_X(\pi) = \dim(X^G)$ for any fin. dim. G-repn X.

Now to compute $\chi_{V \otimes W^*}(\pi) = \dim(V \otimes W^*)^G$

observe that $\left(\text{elements of } V \otimes W^*\right) \Leftrightarrow \left(\text{linear maps } W \to V\right)$

(identify $\sum_i v_i \otimes f_i$ with map $w \mapsto \sum_i f_i(w) v_i$) and

$\left(\text{G-invariant elements of } V \otimes W^*\right)$ correspond to $\left(\text{elements of } \mathrm{Hom}_G(W,V)\right)$

Thus $\dim (V \otimes W^*)^G = \dim \mathrm{Hom}_G(W,V)$. $\square$

Actually $(V \otimes W^*)^G \cong \mathrm{Hom}_G(W,V)$ as vector spaces

(should check this!)

For $g \in G$ let $Z_g = \{h \in G \mid gh = hg\}$ (centralizer subgroup of g.)

<u>Fact</u> Size of $X_g \overset{\text{def}}{=} \{xgx^{-1} \mid x \in G\}$ is $\dfrac{|G|}{|Z_g|}$

Thm Let $g, h \in G$. Then

$$\sum_{\psi \in \mathrm{Irr}(G)} \psi(g)\overline{\psi(h)} = \begin{cases} |Z_g| & \text{if } \chi_g = \chi_h \\ 0 & \text{otherwise.} \end{cases}$$

Pf sketch Since $\mathbb{C}[G]$ is semisimple, we have

① $\mathbb{C}[G] \cong \bigoplus_{\psi \in \mathrm{Irr}(G)} \mathrm{End}(V_\psi)$ where V_ψ is an irred. G-repn with character ψ.

② Can identify $\mathrm{End}(V) \cong V \otimes V^*$

Now interpret $\sum_{\psi \in \mathrm{Irr}(G)} \underbrace{\psi(g)\overline{\psi(h)}}_{\psi(g)\psi(h^{-1})}$ as trace of $\left(\begin{array}{l}\text{linear map } \mathbb{C}[G] \to \mathbb{C}[G] \\ \text{with } x \mapsto g x h^{-1} \\ \text{for } x \in G\end{array}\right)$

This trace is just $\#\{x \in G \mid x = gxh^{-1}\}$

$$= \#\{x \in G \mid g = xhx^{-1}\} = \begin{cases} 0 & \text{if } g, h \text{ not conjugate} \\ |Z_g| & \text{otherwise.} \end{cases} \quad \square$$

<u>Unitary repns</u> A finite dim repn (V, ρ) of a group G (over $k = \mathbb{C}$) is <u>unitary</u> if there is a positive definite Hermitian form $(\cdot, \cdot) : V \times V \to \mathbb{C}$ with

$$(\rho(g)x, \rho(g)y) = (x, y) \quad \forall x, y \in V, \ g \in G$$

<u>Prop</u> If $|G| < \infty$ then any finite dim. G-repn is unitary.

<u>Pf</u> Pick any basis $[v_i]$ for V.

Consider the positive def. Hermitian form on V with

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then the form $(x, y) \overset{\text{def}}{=} \sum_{g \in G} \langle \rho_V(g) x, \rho_V(g) y \rangle$

makes V unitary. $\square$

Prop If (V,ρ) is a fin. dim. unitary repn of (not necessarily finite) group G, then (V,ρ) is semisimple / completely reducible (direct sum of irreducible subrepns)

Pf Choose an irreducible subrepn U of V.

If $U \neq V$ then let $U^\perp = \{v \in V \mid (u,v) = 0 \ \forall u \in U\}$

Then $V = U \oplus U^\perp$ and both $U, U^\perp$ are subrepns, so result follows by induction on dimension. $\square$

<u>Matrix elements</u> Assume G is a finite group and $k = \mathbb{C}$ (as we have been doing so far).

Let (V, ρ_V) be an irred. G-repn.

Choose a positive def. Hermitian form $(\cdot,\cdot)$ on V that is G-invariant, making (V, ρ_V) unitary.

Let $\{v_i\}_{i \in I}$ be an orthonormal basis of V

ie with $(v_i, v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ $\forall\, i, j \in I$

Define $t^V_{ij}(g) = (\rho_V(g) v_i, v_j) = \left(\begin{array}{l}\text{entry in position } (i,j) \text{ of} \\ \text{matrix representing } \rho_V(g) \\ \text{in basis } \{v_i\}_{i \in I} \text{ of } V\end{array}\right)$

(for $g \in G$)

Each t^V_{ij} is a map $G \to \mathbb{C}$, called a <u>matrix elem.</u>

<u>Prop.</u> The rescaled matrix elements

$$\frac{1}{\sqrt{\dim V}}\, t^V_{ij} : G \to \mathbb{C}$$

(as V ranges over all isomorphism classes of irred. G-repns and i, j range over the indices of an orthonormal basis of V) give an orthonormal basis of the space of <u>all</u> functions $G \to \mathbb{C}$ (for the form

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)} \Big)$$

Pf (See textbook)

Note that # of such matrix elements is $\sum_V (\dim V)^2 = |G|$

Character tables

Suppose G is a finite group.

Choose representatives $1 = g_1, g_2, \dots, g_r$ for distinct conjugacy classes in G. Suppose

$\underbrace{\mathbb{1}}_{\text{the character with } g \mapsto 1} = \chi_1, \chi_2, \dots, \chi_r$ are the elements of $\mathrm{Irr}(G)$

Then everything you want to know about $\mathrm{Irr}(G)$ is encoded by the matrix

$\mathrm{Irr}(G)$	$1=g_1$	g_2	$\cdots$	g_r
$\mathbb{1}=\chi_1$	1	1		1
χ_2	$\chi_2(1)$	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$			$\vdots$
χ_r	$\chi_r(1)$	$\chi_r(g_r)$	$\cdots$	$\chi_r(g_r)$

called the <u>Character table</u> of G

<u>Ex</u> If $G = S_3$ symmetric group on 3 letters, then the character table is

$\mathrm{Irr}(S_3)$	1	$(1\ 2)$	$(1\ 2\ 3)$
$\chi_{\ydiagram{3}}$	1	1	1
$\chi_{\ydiagram{2,1}}$	2	0	-1
$\chi_{\ydiagram{1,1,1}}$	1	-1	1

Using the character table + orthogonality relations from today, you can compute sizes of all conjugacy classes / centralizer subgroups in G, and decompose products of characters into irreducibles.