Math 5112 - Lecture # 14

Math S112 - Lecture #14

Last time:

① Frobenius divisibility thm: The dimension of any irreducible complex representation of a finite group divides the size of the group.

② A group G is solvable if there are normal subgroups $\{1\} = G_1 \triangleleft G_2 \triangleleft G_3 \triangleleft \cdots \triangleleft G_n = G$ such that each quotient G_{i+1}/G_i is abelian.

Burnside's thm If G is finite with size $|G| = p^a q^b$ for distinct primes $p, q > 1$ and integers $a, b \geq 0$, then G is solvable.

<u>Pf</u> Assume G is a finite group of order $p^a q^b$ that is not solvable. (Argue by contradiction.)

Assume G is smallest group with these properties. Then G must be simple (since otherwise any proper nontrivial subgroup $N \lhd G$ would be solvable, as would the quotient G/N, implying G is itself solvable)

Last time we proved: <u>Thm</u> Any group with a conjugacy class of size p^k where $p>1$ is prime and $k>0$, is <u>not</u> simple.

Hence G has no conjugacy classes of size p^k or q^k for $k>0$. The size of any conj. class divides $|G|$, so each of these sizes must be 1 or a multiple of pq.

But $p^a q^b = |G| = \sum_{\mathcal{K} \text{ conj class of } G} |\mathcal{K}|$

$$= 1 + \sum_{\substack{\mathcal{K} \text{ nontrivial} \\ \text{conj. class of } G}} |\mathcal{K}|$$

So must have at least one nontrivial conj. class $\{1\} \neq \mathcal{K}$ with $|\mathcal{K}| = 1$ (otherwise we would have $p^a q^b \equiv 1 \pmod{pq}$ which is impossible as $G \neq 1$)

This means center $Z(G) = \{ g \in G \mid gx = xg \ \forall x \in G \}$ has size at least two. As $\{1\} \neq Z(G) \trianglelefteq G$, must have $G = Z(G)$ as G is simple, but then G is solvable, (and abelian) a contradiction. $\square$

Representations of product groups

Recall that if A and B are algebras (over some field k) then the vector space $A \otimes B$ is naturally an algebra.

If V is an A-repn and W is a B-repn then the vector space $V \otimes W$ is naturally an $A \otimes B$-repn.

Moreover, the operation $(V, W) \longmapsto V \otimes W$ is a bijection

$$\left\{ \begin{matrix} \text{finite dim.} \\ \text{irreducible} \\ \text{repns of } A \end{matrix} \right\} \times \left\{ \begin{matrix} \text{finite dim.} \\ \text{irreducible} \\ \text{repns of } B \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{finite dim.} \\ \text{irreducible} \\ \text{repns of } A \otimes B \end{matrix} \right\}$$

(We proved this in an earlier lecture.)

For groups G and H, the <u>(direct) product group</u> $G \times H$ is the set $\{(g,h) \mid g \in G, h \in H\}$ with group product $(g_1, h_1)(g_2, h_2) \overset{\text{def}}{=} (g_1 g_2, h_1 h_2)$.

Over any field k, there is an obvious isomorphism

$$k[G \times H] \cong k[G] \otimes k[H]$$
$$(g,h) \longmapsto g \otimes h$$

So we can view any $k[G] \otimes k[H]$-repn as an (algebra) repn of $k[G \times H]$ or as a (group) repn of $G \times H$.

Prop. If G and H are finite groups then the operation $(V,W) \mapsto V \otimes W$ is a bijection

$$\left\{ \begin{array}{c} \text{irred. repns} \\ \text{of } G \text{ over } k \end{array} \right\} \times \left\{ \begin{array}{c} \text{irreducible} \\ \text{repns of} \\ H \text{ over } k \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{irreducible} \\ \text{repns of} \\ G \times H \text{ over } k \end{array} \right\}$$

for any field k.

Pf. This is a special case of the statement for algebras because all irr. repns of finite groups are finite-dimensional. $\square$

↳ as the associated group algebras are finite-dimensional

Virtual representations $\rightsquigarrow$ formal linear combinations of reps with coefficients in $\mathbb{Z}$

The (split) Grothendieck group of the category $\mathrm{Rep}_k(G) = \{$finite dim. reps of a finite group G / $k\}$ is the abelian group generated by the formal symbols $[V]$ for $V \in \mathrm{Rep}_k(G)$ subject to relations

$$[V \oplus W] = [V] + [W] \text{ for any } W, V \in \mathrm{Rep}_k(G)$$

$$[V] = [W] \text{ if } V \cong W \text{ as } G\text{-reps.}$$

If $K[G]$ is semisimple ($\Leftrightarrow \mathrm{char}(k) \nmid |G|$)

then the Grothendieck group is a free abelian group with basis $[V_1], [V_2], \ldots, [V_r]$ where $V_1, V_2, \ldots, V_r$ represent distinct isomorphism classes of irreducible G-repns.

In this case the Grothendieck group is $\cong \mathbb{Z}^r$ and each element is given uniquely by

$$\sum_{i=1}^{r} n_i [V_i] \quad \text{where } n_i \in \mathbb{Z}.$$

A <u>virtual repn</u> of G is any element of the Grothendieck group.

Each virtual repn has a well-defined <u>character</u>.

In particular, the character of

$$\sum_{i=1}^{r} n_i [V_i] \text{ is just } \sum_{i=1}^{r} n_i \chi_{V_i} : G \to K$$

<u>Lemma</u> Assume $K = \mathbb{C}$ and W is a virtual repn of G with character χ_W.

If $(\chi_W, \chi_W) = 1$ and $\chi_W(1) > 0$ then $W = [V]$ for some irreducible G-repn $V \in \mathrm{Rep}_{\mathbb{C}}(G)$.

<u>Pf</u> Write $W = \sum_{i=1}^{r} n_i [V_i]$.

Then $\chi_W = \sum_{i=1}^{r} n_i \chi_{V_i}$ and $(\chi_W, \chi_W) = \sum_{i=1}^{r} n_i^2$

If this is 1 then we must have

$W = \pm [V_i]$ for some i

and if $\chi_W(1) = n_i \chi_{V_i}(1) > 0$ then must have

$W = [V_i]$. $\square$

<u>Remark</u> When G is finite and $k[G]$ semisimple,

Grothendieck group $\cong$ additive group of class functions on G.

No meaningful distinction between virtual reps and class fns.

But when G is infinite, or $k[G]$ not semisimple, this identification breaks down.

Restriction and induction

Suppose H is a subgroup of G.

Let (V, ρ) be a G-repn.

<u>Def</u> The restriction of this G-repn is the H-repn

$$\mathrm{Res}_H^G(V,\rho) \overset{\text{def}}{=} (V, \underbrace{\rho|_H}_{\rho: G \to GL(V) \text{ restricted to a map } H \to GL(V)})$$

Often write $\mathrm{Res}_H^G(V)$ if $\rho = \rho_V$ is implicit

There is an adjoint operation called <u>induction</u>:

<u>Def</u>. Let (V,ρ) be an H-repn.

Define $\mathrm{Ind}_H^G(V,\rho) = \mathrm{Ind}_H^G(V)$ to be the set of all maps $f: G \to V$ such that

$$f(hx) = \rho(h) f(x) \quad \forall h \in H, \forall x \in G.$$

Clearly $\mathrm{Ind}_H^G(V,\rho)$ is a vector space.

It is also a repn of G for the action

$$g \cdot f = \left(\begin{array}{c} \text{map } G \to V \text{ given by} \\ x \mapsto f(xg) \end{array} \right)$$

Call this the repn of G induced from V.

for $g \in G$, $f \in \mathrm{Ind}_H^G(V,\rho)$.

<u>Pf</u> Check that $(g \cdot f)(hx) = f(hxg)$

$= f(h(xg)) = \rho(h) f(xg) = \rho(h)\,(g \cdot f)(x)$

so $g \cdot f \in \mathrm{Ind}_H^G(V,\rho)$. Moreover, we have

$(g_1 \cdot (g_2 \cdot f))(x) = ((g_1 g_2) \cdot f)(x) = f(x g_1 g_2)$ for $g_i \in G$, $x \in G$. $\square$

Another construction of $\mathrm{Ind}_H^G(V,\rho)$:

Consider the tensor product

$$K[G] \otimes_{K[H]} V \overset{\text{def}}{=} \left\{ \begin{array}{l} \text{quotient of } \underbrace{K[G] \otimes V}_{\text{vector space tensor product}} \\ \text{by subspace spanned by} \\ \text{all elements} \\ \quad gh \otimes x - g \otimes \rho(h) x \\ \text{for } g \in G,\ h \in H,\ x \in V \end{array} \right\}$$

This space is a G-repn for linear action

$$g_1 \cdot (g_2 \otimes x) = (g_1 g_2) \otimes x \quad \text{for } g_i \in G,\ x \in V$$

If $g_1, g_2, \ldots, g_r$ are representatives for the distinct cosets $G/H = \{gH \mid g \in G\}$ and $v_1, v_2, \ldots, v_s$ is a basis for V, then a basis for $k[G] \otimes_{k[H]} V$ is $\{g_i \otimes v_j\}_{\substack{1 \le i \le r \\ 1 \le j \le s}}$

Here $r = \frac{|G|}{|H|}$ and $s = \dim V$.

Prop $\mathrm{Ind}_H^G (V, \rho) \cong k[G] \otimes_{k[H]} V$ as G-reps whenever G is a finite group.

<u>Pf</u> Check that the map

$$f^{\in \mathrm{Ind}_H^G(V,\rho)} \longmapsto \sum_{x \in G} \underbrace{x}_{\in G} \otimes \underbrace{f(x^{-1})}_{\in V} \in k[G] \otimes_{k[H]} V$$

is an isomorphism. In particular, note that

$$g \cdot \left(\sum_{x \in G} x \otimes f(x^{-1}) \right) = \sum_{x \in G} gx \otimes f(x^{-1})$$

$$= \sum_{x \in G} x \otimes f(x^{-1} g)$$

$$= \sum_{x \in G} x \otimes (g \cdot f)(x^{-1}) \qquad \square$$

Some details left to fill in

Cor $\dim(\mathrm{Ind}_H^G(V,\rho)) = \frac{|G|}{|H|}\dim(V)$

if G is a finite group and $\dim(V) < \infty$.

Pf This is the size of the basis given for

$K[G] \otimes_{K[H]} V$. $\square$

Prop Assume G is finite and $\dim(V) < \infty$.

Let χ be character of (V,ρ) and let

$\mathrm{Ind}_H^G(\chi)$ be character of $\mathrm{Ind}_H^G(V,\rho)$.

Then for each $g \in G$, we have

$$\mathrm{Ind}_H^G(\chi)(g) = \sum_{\substack{i \in \{1,2,\dots,r\} \\ g_i^{-1} g g_i \in H}} \chi(g_i^{-1} g g_i) = \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \chi(xgx^{-1})$$

only holds if $\mathrm{char}(k)$ does not divide $|H|$

$g_1, g_2, \dots, g_r$ are the distinct coset reps of $G/H = \{gH \mid g \in G\}$.

<u>Pf</u>, Consider G acting on $k[G] \otimes_{k[H]} V$

Basis for this space is $\{g_i \otimes v_j\}$

Coefficient of $g_i \otimes v_j$ in $g(g_i \otimes v_j) = (gg_i) \otimes v_j$

is $\begin{cases} 0 & \text{if } gg_i \notin g_iH \\ \text{coeff of } v_j \text{ in } \rho(g_i^{-1}gg_i)v_j & \text{if } gg_i \in g_iH \iff g_i^{-1}gg_i \in H \end{cases}$

as in this case $gg_i \otimes v_j = g_i \otimes \rho(g_i^{-1}gg_i)v_j$

Summing over $j \in [1, 2, \dots, s]$ gives $\begin{cases} 0 & \text{if } g_i^{-1}gg_i \notin H \\ \chi(g_i^{-1}gg_i) & \text{else} \end{cases}$

Summing this over $i \in [1, 2, \dots, r]$ gives the first desired formula for $\mathrm{Ind}_H^G(\chi)(g)$.

If $\mathrm{Char}(k) \nmid |H|$ then $|H|$ is invertible in K and

$$\chi(g_i^{-1} g g_i) = \frac{1}{|H|} \sum_{h \in H} \underbrace{\chi(h^{-1} g_i^{-1} g g_i h)}_{= \chi(g_i^{-1} g g_i)} = \frac{1}{|H|} \sum_{x \in g_i H} \chi(x^{-1} g x)$$

whenever $g_i^{-1} g g_i \in H$. Summing over i gives

$$\mathrm{Ind}_H^G(\chi)(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1} g x \in H}} \chi(x^{-1} g x) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x g x^{-1} \in H}} \chi(x g x^{-1})$$

Since $g_i^{-1} g g_i \in H$ iff $x^{-1} g x \in H$ $\forall x \in g_i H$. $\square$