Math 5112 - Lecture # 15

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Last time:

__Thm__ If $\chi : G \to \mathbb{C}$ is a class function on a finite group G, and $(\chi, \chi) \overset{\text{def}}{=} \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} = 1$ and $\chi(1) > 0$ then χ is the character of an irreducible G-repn.

__Thm__ If G, H are finite groups and k is any field, then for any G-repn V over k and any H-repn W over k, the (vector space) tensor product $V \otimes W$ is naturally a $G \times H$-repn via formula $(g,h) : x \otimes y \mapsto \rho_V(g)x \otimes \rho_W(h)y$

The operation $(V, W) \mapsto V \otimes W$ is a bijection

$$\begin{Bmatrix} \text{irr. repns} \\ \text{of } G \text{ over } k \end{Bmatrix} \times \begin{bmatrix} \text{irr. repns} \\ \text{of } H \text{ over } k \end{bmatrix} \xrightarrow{\sim} \begin{Bmatrix} \text{irr. } G \times H\text{-repns} \\ \text{over } k \end{Bmatrix}$$

If H is a subgroup of a finite group G and V is a G-repn, then we write $\mathrm{Res}_H^G(V)$ to denote V viewed as an H-repn via restriction.

$$\begin{cases} \text{Character of } \mathrm{Res}_H^G(V) \text{ is } \chi_V|_H \text{ (character restricted to } H) \\ \text{Dimension of } \mathrm{Res}_H^G(V) \text{ is } \dim(V) \end{cases}$$

Call $\mathrm{Res}_H^G(V)$ the <u>restriction</u> of V.

Suppose W is an H-repn where $H \subset G$ is a subgroup

Let $\mathrm{Ind}_H^G(W) = \left\{ f : G \to W \mid f(hx) = \rho_W(h) f(x) \ \forall h \in H, x \in G \right\}$

$\cong \underbrace{k[G]}_{\substack{\text{viewed} \\ \text{as } (k[G],\, k[H])\text{-bimodule}}} \otimes_{k[H]} \underbrace{W}_{\substack{\text{viewed as} \\ \text{left-}k[H]\text{-module}}}$ (where k is ambient scalar field)

$\mathrm{Ind}_H^G(W)$ is a G-repn (called the <u>induced repn</u>) for action

$$g \cdot f : x \mapsto f(xg) \quad \text{for } g \in G,\ f \in \mathrm{Ind}_H^G(W).$$

Character of $\mathrm{Ind}_H^G(W)$ is $\chi(g) = \displaystyle\sum_{\substack{i \in \{1,2,\dots,r\} \\ g_i^{-1} g g_i \in H}} \chi_W(g_i^{-1} g g_i)$

where $g_1, g_2, \dots, g_r$ are a complete set of coset representatives for $G/H = \{gH \mid g \in G\}$

If char(k) does not divide $|H|$, then can write this as

$$\chi(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1} g x \in H}} \chi_W(x^{-1} g x) \quad \text{for } g \in G$$

Dimension of $\mathrm{Ind}_H^G(W)$ is $\dfrac{|G|}{|H|} \dim(V)$

Restriction and induction usually do not preserve the property of irreducibility.

Useful property If $N \subset H \subset G$ are subgroups and V is an N-repn then $\mathrm{Ind}_N^G(V) \cong \mathrm{Ind}_H^G \mathrm{Ind}_N^H(V)$.

(Proof is an exercise) Similarly, $\mathrm{Res}_N^G = \mathrm{Res}_N^H \mathrm{Res}_H^G$

Frobenius reciprocity Let G be a finite group with a subgroup H. Suppose V is a G-repn and W is an H-repn (both over same field k). Then

$$\underbrace{\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W))}_{\text{(morphisms of } G\text{-repns)}} \cong \underbrace{\mathrm{Hom}_H(\mathrm{Res}_H^G(V), W)}_{\text{(morphisms of } H\text{-repns)}}$$

as vector spaces. (One can make a more precise claim that there is a "natural" isomorphism between these two vector spaces, in sense that the diagrams corresponding to morphisms $V \to V'$ and/or $W \to W'$ all commute.)

Write $\mathrm{Res}_H^G(\chi_V) \overset{\text{def}}{=} \chi_V|_H$ and $\mathrm{Ind}_H^G(\chi_W)$ for the characters of $\mathrm{Res}_H^G(V)$ and $\mathrm{Ind}_H^G(\chi_W)$.

Cor Assuming $k = \mathbb{C}$, if χ is any character of G and ψ is any character of H, then

$$(\chi, \mathrm{Ind}_H^G(\psi)) = (\mathrm{Res}_H^G(\chi), \psi)$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\mathrm{Ind}_H^G(\psi)(g)} \qquad = \frac{1}{|H|} \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

Pf LHS is $\dim \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W))$ and RHS is $\dim \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W)$ if $\chi = \chi_V$, $\psi = \chi_W$. $\square$

Pf of Frobenius reciprocity

Define $\Phi : \operatorname{Hom}_G(V, \underbrace{\operatorname{Ind}_H^G W}_{\text{a set of maps } G \to W}) \longrightarrow \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$

$$\alpha \longmapsto \underbrace{\Big(\underset{\in V}{v} \longmapsto \overbrace{\underbrace{\alpha(v)}_{\text{a map } G \to W}(1_G)}^{\in W} \Big)}_{\text{a map } V \to W} = \Phi(\alpha)$$

along with $\Psi : \operatorname{Hom}_H(\operatorname{Res}_H^G V, W) \longrightarrow \operatorname{Hom}_G(V, \operatorname{Ind}_H^G W)$

$$\beta \longmapsto \Psi(\beta) = \Bigg(\underbrace{v \longmapsto \left(\begin{array}{c} G \longrightarrow W \\ x \longmapsto \beta(\rho_V(x)v) \end{array} \right)}_{\text{a map } V \to \{\text{maps } G \to W\}} \Bigg)$$

Check that (1) $\Phi(\alpha)$ for $\alpha \in \operatorname{Hom}_G(V, \operatorname{Ind}_H^G W)$ is a H-repn morphism.

Pf $\Phi(\alpha)(\rho_V(h)v) \overset{\text{by def.}}{=} \alpha(\rho_V(h)v)(1_G) \overset{\text{as } \alpha \text{ is morphism of } G\text{-repns}}{=} (h \cdot \alpha(v))(1_G)$

for $h \in H$, $v \in V$

$= \alpha(v)(h \cdot 1_G) \overset{\text{by def. of } \operatorname{Ind}_H^G W \ni \alpha(v)}{=} \rho_W(h)\, \alpha(v)(1_G)$

by def of $h \cdot \alpha(v)$ and since $h \cdot 1_G = 1_G \cdot h$

$\overset{\text{by def.}}{=} \rho_W(h)\, \Phi(\alpha)(v)$ □

Now check ② $\left(\Psi(\beta)(v)\right)(hx) = \rho_W(h)\left(\left(\psi(\beta)(v)\right)(x)\right)$

to confirm that $\Psi(\beta)$ is a linear map $V \to \mathrm{Ind}_H^G W$

for $\beta \in \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$

$v \in V$, $h \in H$, $x \in G$

by def of Ψ since ρ_V is a group homomorphism

Pf $\left(\Psi(\beta)(v)\right)(hx) = \beta\left(\rho_V(hx)v\right) = \beta\left(\rho_V(h)\rho_V(x)v\right)$

as β is a morphism of H-repns

$= \rho_W(h)\,\beta\left(\rho_V(x)v\right)$

by def of Ψ

$= \rho_W(h)\left(\left(\Psi(\beta)(v)\right)(x)\right)$ $\square$

Next check that ③ $\Psi(\beta)$ for $\beta \in \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$ is a morphism of G-repns.

Pf. Suffices to compute

$$\Big(\Psi(\beta)\big(\rho_V(g)v\big)\Big)(x) \overset{\text{by definitions}}{=} \beta\big(\rho_V(x)\rho_V(g)v\big)$$

$$\overset{\text{as } \rho_V \text{ is group homomorphism}}{=} \beta(\rho_V(xg)v)$$

$$\overset{\text{by def}}{=} \big(\Psi(\beta)(v)\big)(xg)$$

$$\overset{\text{by def}}{=} \big(g \cdot \Psi(\beta)(v)\big)(x)$$

$\forall g \in G,\ v \in V,\ x \in G$

and this shows that $\Psi(\beta)(\rho_V(g)v) = (g \cdot \Psi(\beta))(v)$ $\square$

Finally check ④ $\Phi \circ \Psi$ = identity map

Pf $\Big(\Phi(\Psi(\beta))\Big)(v) \overset{\text{by def}}{=} \Big(\Psi(\beta)(v)\Big)(1_G)$

$\overset{\text{by def}}{=} \beta(1_G v) = \beta(v)$

$\Rightarrow \Phi(\Psi(\beta)) = \beta$ $\square$

And check ⑤ $\Psi \circ \Phi$ = identity map

Pf $\Big(\big(\Psi(\Phi(\alpha))\big)(v)\Big)(x) \overset{\text{by def}}{=} \Phi(\alpha)\big(\rho_V(x)v\big)$

$\overset{\text{by def}}{=} \Big(\alpha\big(\rho_V(x)v\big)\Big)(1_G)$

$\overset{\text{by def}}{=} \big(x \cdot \alpha(v)\big)(1_G) = \alpha(v)(x)$

Follows that

$\Psi(\Phi(\alpha)) = \alpha$

$\square$

Combining ①–⑤ shows that Φ and Ψ are well-defined inverse isomorphisms of vector spaces. $\square$

<u>Ex</u> If $H = \{1\}$ and $W = K$ then $\mathrm{Ind}_H^G W \cong K[G]$

This is equivalent (when $K = \mathbb{C}$) to saying that

$$(*) \quad (\chi, \mathrm{Ind}_1^G(\mathbb{1})) = \chi(1) \text{ for all } \chi \in \mathrm{Irr}(G)$$

where $\mathrm{Irr}(G)$ = set of irreducible characters for G, and $\mathbb{1} : \{1\} \to \{1_K\}$ is trivial character. Indeed

$$\left(\mathrm{Res}_1^G(\chi), \mathbb{1}\right) = \frac{1}{1} \sum_{1 \in \{1\}} \chi(1)\, \mathbb{1}(1) = \chi(1) \quad \checkmark$$

so by Frobenius reciprocity $(*)$ holds

Representations of Symmetric groups

Let n be a positive integer

Let $[n] = \{1, 2, 3, \dots, n\}$

Let S_n = (group of bijections $[n] \to [n]$)

call this the <u>symmetric group</u> (on n letters)

A <u>partition</u> of n is a sequence of integers

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$
and $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$

Here k can vary from 1 $(\lambda = (n))$ to n $(\lambda = (1, 1, 1, \dots, 1))$

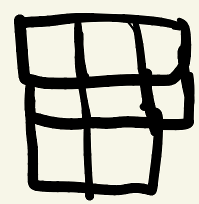
Write $\lambda \vdash n$ to denote that λ is a partition of n

Define $\ell(\lambda) = k$ if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$

Set $\lambda_i = 0$ if $i > \ell(\lambda)$

The <u>(Young) diagram</u> of $\lambda \vdash n$ is the set of positions $D_\lambda = \left\{ (i,j) \in \mathbb{Z} \times \mathbb{Z} \;\middle|\; \begin{matrix} i \geq 1 \\ 1 \leq j \leq \lambda_i \end{matrix} \right\}$

View D_λ as a subset of positions in an $n \times n$ matrix

<u>Ex</u> If $\lambda = (3,3,2)$ then $D_\lambda =$  $= \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \end{pmatrix}$

(Then we can refer to rows/columns/etc of D_λ like in a matrix)

A <u>(standard Young) tableau</u> of shape $\lambda \vdash n$ is a map $T : D_\lambda \longrightarrow [n]$ such that rows are increasing left to right and columns are increasing top to bottom

$T(i,j) < T(i,j+1)$ $\qquad$ $T(i,j) < T(i+1,j)$

<u>Ex</u> If $\lambda = (2,2) \vdash n = 4$ then T could be either

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

Let T_λ be the particular tableau of shape λ whose entries in row i are $\lambda_1 + \lambda_2 + \cdots + \lambda_{i-1} + j$ for $1 \leq j \leq \lambda_i$

Ex If $\begin{cases} \lambda = (4,2,1) \\ n = 7 \end{cases}$ then $T_\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$

Given $\lambda \vdash n$,

define $P_\lambda \overset{\text{def}}{=} \left\{ \begin{array}{l} \text{subset of } \sigma \in S_n \text{ such that if} \\ \sigma(i) = j \text{ then } i \text{ and } j \text{ are in same row of } T_\lambda \end{array} \right\}$

$Q \overset{\text{def}}{=} \left\{ \begin{array}{l} \text{subset of } \sigma \in S_n \text{ such that if} \\ \sigma(i) = j \text{ then } i \text{ and } j \text{ are in same} \\ \text{column of } T_\lambda \end{array} \right\}$

Fact P_λ and Q_λ are both subgroups and $P_\lambda \cap Q_\lambda = \{1\}$.

Finally, define the Young projectors in $\mathbb{Z}[S_n]$ to be

$$a_\lambda \stackrel{\text{def}}{=} \frac{1}{|P_\lambda|} \sum_{g \in P_\lambda} g \in \mathbb{Z}[S_n]$$

$$b_\lambda \stackrel{\text{def}}{=} \frac{1}{|Q_\lambda|} \sum_{g \in Q_\lambda} \operatorname{sgn}(g) g \in \mathbb{Z}[S_n]$$

where $\operatorname{sgn} : S_n \to \{\pm 1\}$ is the sign repn whose value at g is -1 iff g is a product of an odd number of transpositions (i, j).

Let $c_\lambda \stackrel{\text{def}}{=} a_\lambda b_\lambda$ and $V_\lambda \stackrel{\text{def}}{=} \mathbb{C}[S_n] c_\lambda \mathbb{C}[S_n]$

Thm The subspace $V_\lambda \subset \mathbb{C}[S_n]$ is an irreducible S_n-repn (called a Specht module) and each irreducible (complex) repn of S_n is isomorphic to V_λ for a unique partition $\lambda \vdash n$.

Cor All irred. complex repns of S_n are realizable over the rational numbers $\mathbb{Q}$, so have Frobenius-Schur indicator 1.

meaning there exists a basis such that matrices corresponding to all group elems have entries in $\mathbb{Q}$