Math 5112 - Lecture #16

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Last time: Frobenius reciprocity

Let $G \supset H$ be finite groups.

Suppose V is a G-repn and W is an H-repn, both over the same field k.

Then $\dim \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) = \dim \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$

(This statement about dimensions can be rephrased as the existence of a natural isomorphism)

Cor If $k = \mathbb{C}$ then

$$(\chi_V, \mathrm{Ind}_H^G(\chi_W)) = (\mathrm{Res}_H^G(\chi_V), \chi_W)$$

where $(f,g) = \frac{1}{|X|} \sum_{x \in X} f(x)\overline{g(x)}$ for $X = G$ or H

Goal today: Classify irreducible representations of finite symmetric groups over $\mathbb{C}$.

Recall: a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0)$ is a weakly decreasing sequence of positive integers. Set $\ell(\lambda) = k$ and write $\lambda \vdash n$ if $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$.

The <u>(Young) diagram</u> of λ is the set

$$D_\lambda = \left\{ (i,j) \;\middle|\; \begin{matrix} 1 \le i \le \ell(\lambda) \\ 1 \le j \le \lambda_i \end{matrix} \right\}$$

Viewed as a subset of position in a matrix

(so we can refer to rows, columns, diagonals, etc.)

If $\lambda = (4,1,1)$ the $D_\lambda = \begin{array}{|c|c|c|c|} \hline (1,1) & (1,2) & (1,3) & (1,4) \\ \hline (2,1) \\ \cline{1-1} (3,1) \\ \cline{1-1} \end{array}$

A <u>tableau</u> of shape λ is a map $T: D_\lambda \to \mathbb{Z}$

(think of this as a partially filled in matrix)

A tableau T is <u>standard</u> if its entries are the numbers $1, 2, 3, \ldots, n$ for some n, with no repetitions, such that all rows and columns are increasing.

Some examples of standard tableaux:

1	2	3	4
5	6		
7			

1	4	6	7
2	5		
3			

1	2	4	7
3	5		
6			

Call this T_λ for $\lambda = (4,2,1)$

Define T_λ analogously for other partitions λ.

Let n be a positive integer.

Let $[n] = \{1, 2, 3, \ldots, n\}$

Define S_n to be the group of bijections $\sigma : [n] \to [n]$ (i.e., <u>permutations</u> of $[n]$)

Call S_n the <u>symmetric group</u> (on n letters)

For $\lambda \vdash n$, define

$P_\lambda = \{$ subgroup of $\sigma \in S_n$ such that $\sigma(i) = j$ if and only if i and j are in same row of T_λ $\}$

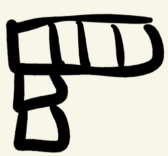
$Q_\lambda = \{$ subgroup of $\sigma \in S_n$ such that $\sigma(i) = j$ if and only if i and j are in same column of $T_\lambda \}$

<u>Facts</u>

$$P_\lambda \cong S_{\lambda_1} \times S_{\lambda_2} \times S_{\lambda_3} \times \cdots \times S_{\lambda_k}$$

$$Q_\lambda \cong S_{\lambda'_1} \times S_{\lambda'_2} \times S_{\lambda'_3} \times \cdots \times S_{\lambda'_p}$$

where λ'_i is number of cells in column i of D_λ and $p = \lambda_1$

$\lambda =$  $= (4,1,1) \rightsquigarrow (\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4) = (3,1,1,1)$

There is a unique nontrivial homomorphism

$$\mathrm{sgn}: S_n \to \{\pm 1\}$$

This map has $\mathrm{sgn}((i_1\ i_2\ i_3 \cdots i_k)) = (-1)^{k-1}$

$$\mathrm{sgn}((i\ j)) = -1$$

Define $c_\lambda = a_\lambda b_\lambda$ for $a_\lambda = \sum_{g \in P_\lambda} g \in \mathbb{Z}[S_n]$

(Note: defns of a_λ, b_λ differ from textbook by constant factor) $b_\lambda = \sum_{g \in Q_\lambda} \mathrm{sgn}(g) g \in \mathbb{Z}[S_n]$

Then let $V_\lambda = \mathbb{C}[S_n] c_\lambda \subset \mathbb{C}[S_n]$

Call this left S_n-module / S_n-repn a <u>Specht module</u>.

(**Note:** $V_\lambda = \mathbb{C}\text{-span}\{\sigma c_\lambda \mid \sigma \in S_n\}$)

Thm Each V_λ for $\lambda \vdash n$ is an irreducible S_n-repn. If V is any irreducible complex S_n-repn then $V \cong V_\lambda$ for a unique $\lambda \vdash n$.

Ex If $\lambda = (n)$ so $\ell(\lambda) = 1$ then $P_\lambda = S_n$ and $Q_\lambda = \{1\}$ so $c_\lambda = \sum_{g \in S_n} g$ and $\sigma c_\lambda = c_\lambda$ $\forall \sigma \in S_n$ and so $V_{(n)} \cong \mathbb{1}$ trivial repn of S_n

Ex If $\lambda = (1,1,1,\dots,1)$ so $\ell(\lambda) = n$ then

$P_\lambda = \{1\}$ and $Q_\lambda = S_n$ so $c_\lambda = \sum_{g \in S_n} \mathrm{sgn}(g) g$

and $\sigma c_\lambda = \mathrm{sgn}(\sigma) c_\lambda \; \forall \sigma \in S_n$ and so

$$V_{(1,1,1,\dots,1)} \cong (\mathbb{C}, \mathrm{sgn}) \text{ sign repn of } S_n.$$

$\mathbb{C}$: vector space

sgn: homomorphism $S_n \to \mathbb{C} \setminus \{0\} = GL(\mathbb{C})$

We will prove the theorem through a sequence of lemmas.

Fact Since $P_\lambda \cap Q_\lambda = \{1\}$, if $p_1, p_2 \in P_\lambda$ and $q_1, q_2 \in Q_\lambda$ and $p_1 q_1 = p_2 q_2$, then $p_1 = p_2$ and $q_1 = q_2$

$\Leftrightarrow p_2^{-1} p_1 = q_2 q_1^{-1} \in P_\lambda \cap Q_\lambda$

Thus any $g \in P_\lambda Q_\lambda$ has a unique factorization $g = pq$ with $p \in P_\lambda$, $q \in Q_\lambda$.

Lemma Suppose $g \in S_n$.

① If $g \in P_\lambda Q_\lambda$ and $g = pq$ for $p \in P_\lambda$, $q \in Q_\lambda$, then

$$a_\lambda g b_\lambda = \operatorname{sgn}(q)\, a_\lambda b_\lambda = \operatorname{sgn}(q)\, c_\lambda$$

② If $g \notin P_\lambda Q_\lambda$ then $a_\lambda g b_\lambda = 0$.

Pf ① is easy: $a_\lambda g b_\lambda = \underbrace{a_\lambda p}_{= a_\lambda} \underbrace{q b_\lambda}_{= \operatorname{sgn}(q) b_\lambda} = \operatorname{sgn}(q)\, C_\lambda$. (assuming $g \in P_\lambda Q_\lambda$)

② is harder. If there exists a transposition $t=(i,j)\in S_n$ where $1\le i<j\le n$ such that $t\in P_\lambda$ and $g^{-1}tg\in Q_\lambda$, then $a_\lambda g b_\lambda = 0$ because

$$a_\lambda g b_\lambda = a_\lambda t g b_\lambda = a_\lambda g \underbrace{g^{-1} t g b_\lambda}_{=-b_\lambda} = -a_\lambda g b_\lambda .$$

It suffices to show if there are no such transpositions then $g\in P_\lambda Q_\lambda$.

Let $T=T_\lambda$ and $T' = gT_\lambda =$ (tableau formed by applying g to each entry of T)

Ex If $g=(134)(67)$ and $T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$ then $T' = \begin{array}{|c|c|c|c|} \hline 3 & 2 & 4 & 1 \\ \hline 5 & 7 & \multicolumn{2}{c}{} \\ \cline{1-2} 6 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$

A transposition $t=(i,j)$ has the properties noted above iff i and j are in same row of T and same column of T'. (Take $t=(5,6)$ in example)

Suppose no such i,j exist. Then any two elements in first row of T belong to different columns of T'.

Hence $\exists\, p_1 \in P_\lambda$ and $q_1' \in g Q_\lambda g^{-1}$ such that $p_1 T$ and $q_1' T'$ have same first row.

Now just repeat this argument for second, third, ..., row
$\rightsquigarrow$ conclude by induction on # of rows that there are
$p \in P_\lambda$ and $q' \in g Q_\lambda g^{-1}$ such that $pT = q'T'$

But this means that $pT = q'gT = g\,qT$

for $q = g^{-1}q'g \in Q_\lambda$

Observe: $g_1 T = g_2 T \Leftrightarrow g_1 = g_2 \in S_n$.

Thus $p = gq \Rightarrow g = pq^{-1} \in P_\lambda Q_\lambda$ ✓ $\square$

The <u>lexicographic order</u> on partitions is the total order with $\lambda > \mu$ iff $\exists j$ such that $\mu_j < \lambda_j$ and $\mu_i = \lambda_i$ for all $1 \leq i < j$, where we set $\mu_i = 0$ for $i > \ell(\mu)$.

Lemma Assume $\lambda, \mu \vdash n$ and $\lambda > \mu$. Then

$$a_\lambda \mathbb{C}[S_n] b_\mu = 0.$$

Pf Suffices to show that for any $g \in S_n$, there exists a transposition $t = (i,j) \in S_n$ with $t \in P_\lambda$ and $g^{-1} t g \in Q_\mu$ as then

$$a_\lambda g b_\mu = a_\lambda t g b_\lambda = a_\lambda g\, g^{-1} t g\, b_\mu = -a_\lambda g b_\mu.$$

Let $T = T_\lambda$ and $T' = g T_\mu$.

Claim: There are numbers $a < b$ appearing in same row of T and same column of T'.

Let j be first index with $\mu_j < \lambda_j$.

So $\mu_i = \lambda_i$ for $1 \leq i < j$.

If $j = 1$ then our claim must hold by pigeonhole principle. If $j > 1$ and any two elements of first row of T are in different columns of T', then we can find $p \in P_\lambda$ and $q' \in g Q_\lambda g^{-1}$ such that pT and $q'T'$ have same first row.

Repeating this argument for second, third, ..., row / by induction on j, conclude that our claim is true.

Now, given claim, the transposition $t = (a,b)$ has the desired properties $\Rightarrow a_\lambda \mathbb{C}[S_n] b_\mu = 0$. $\square$

<u>Lemma</u> $c_\lambda^2 = \frac{n!}{\dim V_\lambda} c_\lambda$. $\rightsquigarrow$ $\boxed{c_\lambda \text{ is proportional to an idempotent.}}$

<u>Pf</u> Easy to see that $c_\lambda^2 = k c_\lambda$ for some $k \in \mathbb{Z} \setminus \{0\}$

Since $c_\lambda^2 = a_\lambda (b_\lambda a_\lambda) b_\lambda$ and $a_\lambda g b_\lambda$ is $\pm a_\lambda b_\lambda$ or 0 for each $g \in S_n$.

So $\left(\frac{1}{k} c_\lambda\right)^2 = \frac{1}{k^2} c_\lambda^2 = \frac{1}{k} c_\lambda$ is an idempotent.

In some basis of $\mathbb{C}[S_n]$, the matrix of $\frac{1}{k} c_\lambda$ acting by left multiplication is the idempotent matrix

$$\dim V_\lambda \left\{ \left[\begin{array}{c|c} \ddots & 0 \\ \hline 0 & 0 \end{array}\right] \right.$$

Thus $\operatorname{trace}\left(\frac{1}{k} c_\lambda\right) = \dim V_\lambda = \frac{1}{k} \operatorname{trace}(c_\lambda)$ and it's easy to see that the trace of c_λ acting by left multiplication on $\mathbb{C}[S_n]$ is $n!$

This means $\frac{1}{k} = \frac{\dim V_\lambda}{\operatorname{trace}(c_\lambda)} = \frac{\dim V_\lambda}{n!} \Rightarrow k = \frac{n!}{\dim V_\lambda}$ $\square$

__Lem__ Suppose A is an algebra with idempotent $e = e^2 \in A$. If M is a left A-module, then $eM \cong \mathrm{Hom}_A(Ae, M)$

$$\begin{array}{rcl} e \in eM & x \longmapsto & f_x : a \mapsto ax \\ ef(e) = f(e^2) = f(e) & \longleftarrow & f \end{array}$$

__Pf__ Can deduce this directly, or by noting that $(1-e)^2 = 1-e$ and $A = Ae \oplus A(1-e)$ and $\mathrm{Hom}_A(A, M) \cong M$. $\square$

Pf of thm Let $\lambda, \mu \vdash n$ with $\lambda \geq \mu$.

Then $\mathrm{Hom}_{S_n}(V_\lambda, V_\mu) = \mathrm{Hom}_{S_n}(\mathbb{C}[S_n]c_\lambda, \mathbb{C}[S_n]c_\mu)$

by previous lemma

$$\overset{\checkmark}{=} c_\lambda \mathbb{C}[S_n] c_\mu = \begin{cases} 0 & \text{if } \lambda > \mu \text{ (by lemma)} \\ \mathbb{C}c_\lambda & \text{if } \lambda = \mu \end{cases}$$

So this Hom-space is 1-dimensional if $\lambda = \mu$ and zero otherwise. By Schur's lemma, this means V_λ is irreducible and $V_\lambda \ncong V_\mu$ if $\lambda \neq \mu$.

Finally, the V_λ's give all isomorphism classes of irr. complex S_n-reps because # partitions of n is # conj. classes of S_n

which is # isomorphism classes of irred. reps / $\mathbb{C}$. $\square$