Math 5112 - Lecture #19

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Last time: Schur-Weyl duality

* This is a natural correspondence between irred. repns of symmetric groups and general linear groups that explains why both families are indexed by partitions and have similar character formulas

* Such a duality exists for any pair of commuting algebras $A, B \subset End(V)$

commuting means
$$A = \{ a \in End(V) \mid ab = ba \ \forall b \in B \}$$
$$B = \{ b \in End(V) \mid ab = ba \ \forall a \in A \}$$

Schur-Weyl duality concretely:

Pick a nonzero vector space V def'd over $\mathbb{C}$

Choose a positive integer n.

① Then S_n acts on $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$ by permuting factors.

② $\mathfrak{gl}(V) = $ (Lie algebra $\mathrm{End}(V)$) acts "diagonally" on $V^{\otimes n}$: $g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$ for $g \in \mathfrak{gl}(V)$

① extends to a repn of $\mathbb{C}[S_n]$ on $V^{\otimes n}$

② extends to a repn of $\mathcal{U}(\mathfrak{gl}(V))$ on $V^{\otimes n}$

Let A and B be the images of $\mathbb{C}[S_n]$ and $\mathcal{U}(\mathfrak{gl}(V))$ in $\mathrm{End}(V^{\otimes n})$

Then Schur Weyl duality refers to the following properties:

Fact A and B are commuting algebras of each other in $\mathrm{End}(V^{\otimes n})$

Fact A and B are both semisimple, and $V^{\otimes n}$ is a semisimple $A\otimes B$-repn, and therefore a semisimple $\mathbb{C}[S_n]\otimes \mathcal{U}(\mathfrak{gl}(V))$-repn

[Note: A and B are both finite-dim. since $\mathrm{End}(V^{\otimes n})$ is finite-dim, but $\mathcal{U}(\mathfrak{gl}(V))$ is infinite-dim.]

Fact As a $\mathbb{C}[S_n]\otimes \mathcal{U}(\mathfrak{gl}(V))$-repn,

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes L_\lambda$$ where V_λ is the Specht module of S_n, and L_λ is zero or an irreducible $\mathcal{U}(\mathfrak{gl}(V))$-repn.

Moreover, in this decomposition, we have $L_\lambda \not\cong L_\mu$ if $L_\lambda \neq 0$ and $L_\mu \neq 0$ and $\lambda \neq \mu$.

Fact Each nonzero L_λ is also irreducible as a repn of $GL(V) = (\text{invertible elements of } \mathfrak{gl}(V))$.

Schur polynomials

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ be a partition of n.

Set $\lambda_i = 0$ for $i > k$ and define $\ell(\lambda) = k$

Choose integer $N \geq \ell(\lambda)$.

Let $\Delta(x_1, x_2, \dots, x_N) = \prod_{1 \le i < j \le N} (x_i - x_j) = \det\left[x_i^{N-j}\right]_{1 \le i,j \le N}$

Let $\Delta_\lambda(x_1, x_2, \dots, x_N) = \det\left[x_i^{N-j+\lambda_j}\right]_{1 \le i,j \le N}$

Def The <u>Schur polynomial</u> of λ is the quotient

$$s_\lambda(x_1, x_2, \dots, x_N) = \frac{\Delta_\lambda(x_1, x_2, \dots, x_N)}{\Delta(x_1, x_2, \dots, x_N)}$$

Claim Each $s_\lambda(x_1, \dots, x_N)$ is a polynomial that is symmetric in the x_i variables, meaning if we reorder the variables the polynomial is unchanged.

<u>Pf</u> $\Delta(x_1, x_2, \dots, x_N)$ divides $\Delta_\lambda(x_1, x_2, \dots, x_N)$

since each factor $x_i - x_j$ divides since setting $x_i = x_j$ in gives zero (as then we're taking det of a matrix with two equal rows).

Symmetry of s_λ follows by noting that reordering the variables $x_1, x_2, \dots, x_N$ multiplies the value of $\Delta(x)$ and $\Delta_\lambda(x)$ by ± 1 (same factor for each det)

These factors cancel and so

$$s_\lambda(x_{i_1}, x_{i_2}, \dots, x_{i_N}) = s_\lambda(x_1, x_2, \dots, x_N)$$

for any $\{i_1, i_2, i_3, \dots, i_N\} = \{1, 2, 3, \dots, N\}$ $\square$

Ex Suppose $\lambda = (3)$ so $n=3$, $k=\ell(\lambda)=1$

Take $N=2$. Then

$$s_{(3)}(x_1, x_2) = \frac{\det\left[x_i^{N-j+\lambda_j}\right]}{\det\left[x_i^{N-j}\right]} = \frac{\det\begin{bmatrix} x_1^4 & 1 \\ x_2^4 & 1 \end{bmatrix}}{\det\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix}} = \frac{x_1^4 - x_2^4}{x_1 - x_2}$$

$T = \boxed{i_1}\boxed{i_2}\cdots\boxed{i_n}$

$$= x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$$ ← symmetric in x_1 and x_2

$$= x_1 x_1 x_1 + x_1 x_1 x_2 + x_1 x_2 x_2 + x_2 x_2 x_2$$

In general, $$s_{(n)}(x_1, x_2, \ldots, x_N) = \sum_{1 \le i_1 \le i_2 \le i_3 \le \cdots \le i_n \le N} x_{i_1} x_{i_2} \cdots x_{i_n}$$

Ex Suppose $\lambda = (1,1,1)$, $k = \ell(\lambda) = 3$ and $n = 3$

Let $N = 3$. Then

$$s_{(1,1,1)}(x_1,x_2,x_3) = \frac{\det\begin{bmatrix} x_1^3 & x_1^2 & x_1 \\ x_2^3 & x_2^2 & x_2 \\ x_3^3 & x_3^2 & x_3 \end{bmatrix}}{\det\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix}} = x_1 x_2 x_3$$

$$T = \begin{array}{|c|} \hline i_1 \\ \hline i_2 \\ \hline i_3 \\ \hline i_4 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array}$$

In general, $$s_{(1,1,1,\dots,1)}(x_1,x_2,\dots,x_N) = \sum_{1 \le i_1 < i_2 < i_3 < \dots < i_n \le N} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$\prod_i x_i^{\#(\text{entries of } T \text{ equal to } i)}$

General formula: $$s_\lambda(x_1,x_2,\dots,x_N) = \sum_T x^T$$

where T varies over all "semistandard" tableaux of shape λ with entries contained in $\{1,2,3,\dots,N\}$.

↑ rows weakly increasing, columns strictly increasing

Consequence of Schur-Weyl duality:

If $i = (i_1, i_2, i_3, \ldots)$ is a sequence of nonnegative integers with $\sum_m m \cdot i_m = n$ and $c_i \in S_n$ is a permutation with i_m cycles of size m, then

$$\prod_{m \geq 1} \underbrace{(x_1^m + x_2^m + \cdots + x_N^m)^{i_m}}_{\text{"power-sum" symmetric polynomial}} = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq N}} \underbrace{\chi_\lambda(c_i)}_{\text{character of the Specht module } V_\lambda} s_\lambda(x_1, x_2, \ldots, x_N)$$

(For a detailed proof, see textbook)

Recall from Schur-Weyl duality that we have certain $GL(V)$-repns L_λ indexed by partitions.

Assume $V = \mathbb{C}^N$ so $GL(V) = GL_N(\mathbb{C})$

Let $g \in GL_N(\mathbb{C})$ and suppose its eigenvalues are $x_1, x_2, \dots, x_N$

<u>Thm</u> ("Weyl character formula") $\Leftrightarrow$ Schur polynomials are characters of irreducible $GL_n(\mathbb{C})$-repns.

The $GL_n(\mathbb{C})$-repn L_λ is nonzero (and therefore irreducible) if and only if $N \geq \ell(\lambda)$, in which case the value of its character at g is $s_\lambda(x_1, x_2, \dots, x_N) \in \mathbb{C}$. In particular,

$$\dim L_\lambda = s_\lambda(1,1,1,\dots,1) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j-i} \in \mathbb{N}$$

(when $\ell(\lambda) \leq N$)

Pf See textbook. Idea is to compute trace of

$\underbrace{g^{\otimes n}}_{\in \mathrm{End}(V^{\otimes n})} \underbrace{c_i}_{\in S_n}$ acting on $V^{\otimes n}$ and note that this is equal to

$$\sum_{\lambda} \chi_{\lambda}(c_i)\, \mathrm{Tr}_{L_\lambda}(g)$$

by Schur Weyl duality. Dimension follows from a more general algebraic identity discussed in textbook. $\square$

Thm These representations L_λ for partitions λ with $\ell(\lambda) \leq N$ give all irreducible polynomial representations of $GL_n(\mathbb{C})$

where a polynomial repn is a finite-dim complex repn (V, ρ) where in some basis of V, the matrix of $\rho(g)$ has the form $[\rho_{ij}(g)]_{1 \leq i,j \leq \dim V}$ where each $\rho_{ij}(g)$ is a polynomial function of the entries of g and $1/\det g$ (ρ_{ij} does not depend on g)

Pf See textbook. Each L_λ is a polynomial repn because its a subrepn of $(\mathbb{C}^N)^{\otimes n}$ which is a polynomial repn. $\square$

Miscellaneous : Artin's theorem

Thm Let X be a set of subgroups of a finite group G, such that if $H \in X$ then $gHg^{-1} \in X$ for all $g \in G$. The following two properties are equivalent:

① Any $g \in G$ belongs to some $H \in X$

② Any irreducible complex character $\psi \in \mathrm{Irr}(G)$ belongs to $\mathbb{Q}$-span $\left\{ \mathrm{Ind}_H^G(\phi) \,\middle|\, \begin{array}{l} \phi \in \mathrm{Irr}(H) \\ H \in X \end{array} \right\}$

Pf If $g \in G$ has $g \notin H$ for all $H \in X$

then $xgx^{-1} \notin H$ for all $H \in X$ (as otherwise $g \in x^{-1}Hx \in X$)

so if $H \in X$ and $\phi \in Irr(H)$ then

$$\mathrm{Ind}_H^G(\phi)(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \phi(xgx^{-1}) = 0 \quad \forall\, H \in X,\ \phi \in Irr(H)$$

sum is empty by assumption

Thus, if ② holds, and $g \in G$ does not belong to any subgroup in X, then $\psi(g) = 0$ for all $\psi \in Irr(G)$.

This is impossible: the trivial character of G takes value 1 at all $g \in G$. Hence if ② holds then ① must hold.

To show ① $\Rightarrow$ ②: let f be a class function $G \to \mathbb{C}$ with $(f, \mathrm{Ind}_H^G(\phi)) = 0$ for all $\phi \in \mathrm{Irr}(H)$, $H \in X$

By Frobenius reciprocity, $(\mathrm{Res}_H^G(f), \phi) = 0$ $\forall \phi \in \mathrm{Irr}(H)$, $H \in X$, so f must vanish on H for every H in X. Assuming ①, this means that $f = 0$.

Using Gram-Schmidt process, construct an orthogonal basis of $\mathbb{Q}\text{-span}\left\{ \mathrm{Ind}_H^G(\phi) \mid \phi \in \mathrm{Irr}(H), H \in X \right\}$, say $\psi_1, \psi_2, \ldots, \psi_k$

Then any $\tau \in \mathrm{Irr}(G)$ has

$$\tau = \frac{(\tau, \psi_1)}{(\psi_1, \psi_1)} \psi_1 + \frac{(\tau, \psi_2)}{(\psi_2, \psi_2)} \psi_2 + \dots + \frac{(\tau, \psi_k)}{(\psi_k, \psi_k)} \psi_k$$

Since if Δ is difference of the two sides, then $(\Delta, \mathrm{Ind}_H^G(\phi)) = 0$ $\forall \phi \in \mathrm{Irr}(H)$, $H \in X$, so $\Delta = 0$.

Thus ① $\Rightarrow$ ②. $\square$

Cor Any irreducible complex character of a finite group is a rational linear combination of irreducible characters induced from cyclic subgroups.

pf Take $X = \{ \langle g \rangle \mid g \in G \}$ satisfies ①, hence ②. $\square$