Math 5112 - Lecture #20

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Last time:

(Precise form of Schur-Weyl duality)

As a $S_n \times GL_N(\mathbb{C})$ representation,

$$(\mathbb{C}^N)^{\otimes n} \cong \bigoplus_{\substack{\lambda \text{ partition of } n \\ \ell(\lambda) \leq N}} V_\lambda \otimes L_\lambda$$

S_n permutes tensor factors

$GL_N(\mathbb{C})$ multiplies with all tensor factors

$\ell(\lambda)$ is number of nonzero parts of $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots > 0)$

where V_λ is the usual Specht module of S_n and L_λ is a (nonzero) irreducible $GL_N(\mathbb{C})$-repn

Moreover, we have $V_\lambda \not\cong V_\mu$ and $L_\lambda \not\cong L_\mu$ if $\lambda \neq \mu$ for partitions λ, μ of n with $\ell(\lambda) \leq N$, $\ell(\mu) \leq N$.

__Thm__ The $GL_N(\mathbb{C})$-repns L_λ, as λ varies over all partitions (of nonnegative integers) with at most N nonzero parts, give all irreducible __polynomial repns__ of $GL_N(\mathbb{C})$ up to isomorphism.

$\hookrightarrow$ means there is a basis of the repn such that we can write down the action of any $g \in GL_N(\mathbb{C})$ as a matrix whose entries are polynomials in entries of g and g^{-1}

<u>Thm</u> (Weyl character formula)

If $g \in GL_N(\mathbb{C})$ with eigenvalues $x_1, x_2, \ldots, x_N \in \mathbb{C}$ (repeated with multiplicity) then the character of L_λ evaluated at g is the value of the <u>Schur polynomial</u>

$$s_\lambda(x_1, x_2, \ldots, x_N) \stackrel{\text{def}}{=} \frac{\det\left[x_i^{N-j+\lambda_j}\right]_{i,j}}{\det\left[x_i^{N-j}\right]_{i,j}}$$

(here $1 \le i,j \le N$)

Also: <u>Artin's thm</u>. Let X be a set of subgroups of a finite group G such that if $H \in X$, $g \in G$ then $gHg^{-1} \in X$. Then TFAE: ① $G = \bigcup_{H \in X} H$ and ② $\mathrm{Irr}(G) \subset \mathbb{Q}\text{-span}\{\mathrm{Ind}_H^G \phi \mid \phi \in \mathrm{Irr}(H), H \in X\}$

Plan for next three weeks:

1. (Today) quick survey of quiver repn theory and Gabriel's theorem (skipping proofs)

2. Lectures 21, 22, 23: category theory and homological algebra

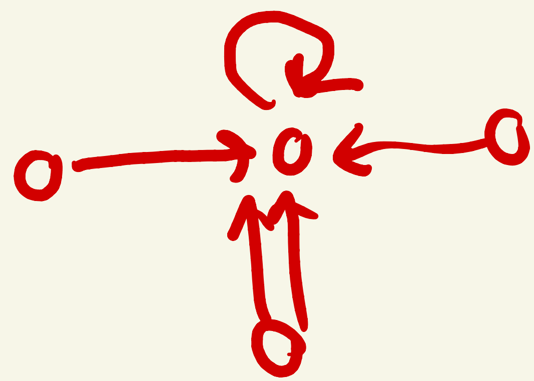
3. Lectures 24-25: repn theory for non-semisimple finite-dim algebras

In addition to HW6, (probably) one or two more HW assignments

(No final exam).

Recall that a <u>quiver</u> $Q = (\overset{\text{vertices}}{V}, \overset{\text{edges}}{E})$ is a directed graph (with multiple edges between two vertices and self loops allowed).

<u>Ex</u>



A <u>representation</u> of the quiver Q is an assignment to each vertex $i \in V$ a vector space X_i and to each edge $(i \to j) \in E$ a linear map $f_{i \to j} : X_i \to X_j$

The (more) interesting thing to consider is isomorphism classes of quiver repns.

If $(X_\bullet, f_\bullet)$ and $(Y_\bullet, g_\bullet)$ are representations of the same quiver $Q = (V, E)$, then a <u>morphism</u> $\phi : (X_\bullet, f_\bullet) \to (Y_\bullet, g_\bullet)$ is a collection of linear maps $\phi_i : X_i \to Y_i$ for $i \in V$ such that the diagram

$$\begin{tikzcd} X_i \arrow[r, "\phi_i"] \arrow[d, "f_{i\to j}"'] & Y_i \arrow[d, "g_{i\to j}"] \\ X_j \arrow[r, "\phi_j"] & Y_j \end{tikzcd}$$

commutes for all edges $(i \to j) \in E$.

Such a morphism is an isomorphism if each ϕ_i is an isomorphism of vector spaces.

There is a natural way of composing morphisms of quiver reps, as well as an obvious identity morphism. Being an isomorphism is equivalent to having a left and right inverse.

Fact Quiver reps (for a fixed quiver Q) with this notion of morphism are the same thing as algebra reps (with usual morphisms) for a certain path algebra which has the set of all directed paths in Q as a basis.

There is also a notion of direct sum for quiver representations. A quiver repn is <u>indecomposable</u> if it is nonzero and not equal to the direct sum of two nonzero quiver repns.

<u>Ex</u> Suppose Q has one vertex, no edges: $\bullet$

A repn of Q is just a choice of a vector space for this vertex (no other data).

A repn of Q is indecomposable iff this vector space has dim $=1$.

Ex Suppose Q is $\bullet \longrightarrow \bullet$ (two vertices, one edge)

A repn of Q is a choice of linear map between arbitrary vector spaces $\left(V \xrightarrow{f} W \right)$

An isomorphism of quiver repns $(V \xrightarrow{f} W) \cong (V' \xrightarrow{f'} W')$ is a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\sim"] \arrow[d, "f"'] & V' \arrow[d, "f'"] \\ W \arrow[r, "\sim"] & W' \end{tikzcd}$$

Claim: determining the isomorphism class of $(V \xrightarrow{f} W)$ as quiver repn is equivalent to doing Gaussian elimination on the matrix of f in some bases of V and W.

We can always decompose quiver repn $(V \xrightarrow{f} W)$ as

$$(V \xrightarrow{f} W) = \left(\ker(f) \xrightarrow{0} 0\right) \oplus \left(V/\ker(f) \xrightarrow[\cong]{f} \mathrm{image}(f)\right)$$
$$\oplus \left(0 \xrightarrow{0} W/\mathrm{image}(f)\right)$$

If $\dim(\ker f) = n$ then $(\ker(f) \to 0) \cong (\mathbb{K} \to 0)^{\oplus n}$

as quiver repns, assuming ambient field is $\mathbb{K}$.

If $\dim(W/\mathrm{image}(f)) = m$ then $(0 \to W/\mathrm{image}(f))$

$\cong (0 \to \mathbb{K})^{\oplus m}$

always holds

If $\dim(V/\ker f) = \dim(\mathrm{image}(f)) = n$

then $\left((V/\ker f) \xrightarrow{f} \mathrm{image}(f)\right) \cong (\mathbb{K} \xrightarrow{id} \mathbb{K})^{\oplus n}$

Thus $Q = (\bullet \longrightarrow \bullet)$ has three indecomposable repns: $(0 \longrightarrow \mathbb{k})$, $(\mathbb{k} \longrightarrow 0)$, and $(\mathbb{k} \xrightarrow{id} \mathbb{k})$

<u>Ex</u> The quiver $\bullet \longrightarrow \bullet \longrightarrow \bullet$ has six indecomposable repns given by

$$\mathbb{k} \to 0 \to 0 \qquad 0 \to 0 \to \mathbb{k} \qquad \mathbb{k} \xrightarrow{id} \mathbb{k} \to 0$$
$$\mathbb{k} \xrightarrow{id} \mathbb{k} \xrightarrow{id} \mathbb{k} \qquad 0 \to \mathbb{k} \xrightarrow{id} \mathbb{k} \qquad 0 \to \mathbb{k} \to 0$$

The quiver $\bullet \longrightarrow \bullet \longleftarrow \bullet$ also has six indecomposable repns (same as above, changing orientation of arrows)

Let $\Gamma = (V, E)$ be an (undirected) graph with vertices V and edges E, multiples edges between two vertices allowed, but no self-loops.

Assume V is finite and number these vertices as $1, 2, 3, \ldots, N$. The <u>adjacency matrix</u> of Γ is the $N \times N$ matrix $R_\Gamma = [r_{ij}]$ where r_{ij} is number of edges between vertex i and j.

The <u>Cartan matrix</u> of Γ is $A_\Gamma = 2I - R_\Gamma$

Def Γ is a Dynkin diagram if Γ is connected and if the quadratic form on $\mathbb{R}^N$

$$(v,w) \overset{\text{def}}{=} v^T A_\Gamma w \in \mathbb{R} \quad (\text{for } v,w \in \mathbb{R}^N)$$

is positive definite (meaning $(v,v) > 0$ for all $0 \neq v \in \mathbb{R}^N$)

__Ex__ The graphs of type A_N, D_N, and E_N are

A_N: $1 - 2 - 3 - \cdots - N$

D_N:

$$
\begin{tikzcd}[arrows=dash]
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & (N-2) \arrow[r] \arrow[d] & (N-1) \\
 & & & & N &
\end{tikzcd}
$$

(Note: $A_3 = D_3$)

E_N:

$$
\begin{tikzcd}[arrows=dash]
1 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] \arrow[d] & 5 \arrow[r] & 6 \arrow[r] & \cdots \arrow[r] & N \\
 & & 2 & & & &
\end{tikzcd}
$$

(Note: $D_5 = E_5$)

__HW Ex__: A_N, D_N, E_6, E_7, E_8 are Dynkin diagrams.

Thm Γ is a Dynkin diagram if and only if (with its vertices labeled in some order) it coincides with one of the graphs A_N $(N \geq 1)$, D_N $(N \geq 4)$ or E_N (for $N \in \{6,7,8\}$).

A quiver is of finite type if it has finitely many isomorphism classes of indecomposable representations.

<u>Gabriel's thm</u> A connected quiver is of finite type if and only the undirected graph obtained by ignoring edge orientations is a Dynkin diagram

[Ex.: we saw above that the quivers whose associated undirected graphs are A_1, A_2, or A_3 are all of finite type]

When a quiver $Q = (V, E)$ is of finite type, any given indecomposable repn $(X_\bullet, f_\bullet)$ is uniquely determined by its <u>dimension vector</u> (defined to be the map $V \to \mathbb{N}$, $i \mapsto \dim X_i$) (up to isomorphism)

Moreover, the dimension vectors that can occur are in bijection with the positive roots of the associated Dynkin diagram.

This explains why the quivers $\bullet \to \bullet \to \bullet$ and $\bullet \to \bullet \leftarrow \bullet$ both have 6 indecomposable repns up to $\cong$. The positive roots of the type A_N Dynkin diagram are $\Phi^+_{A_N} = \{ e_i - e_j \mid 1 \le i < j \le N+1 \}$

so $|\Phi^+_{A_N}| = \binom{N+1}{2}$ which is 6 for $N=3$