Math 5112 - Lecture #21

Math S112 - Lecture #21

Last time: Gabriel's theorem

A quiver $Q = (V, E)$ is a directed graph, with multiple edges and self-loops allowed.

A quiver repn (for a given Q) is an assignment of a vector space X_i to each vertex $i \in V$ and a linear map $f_{ij} : X_i \to X_j$ for each edge $i \to j$

Two quiver rep'ns $(X_\bullet, f_\bullet)$ and $(Y_\bullet, g_\bullet)$ for the quiver Q are isomorphic if there are linear bijections $\phi_i : X_i \to Y_i$ for all vertices i such that every diagram

$$\begin{tikzcd} X_i \arrow[r, "\phi_i"] \arrow[d, "f_{ij}"'] & Y_i \arrow[d, "g_{ij}"] \\ X_j \arrow[r, "\phi_j"] & Y_j \end{tikzcd}$$

commutes (for every edge $i \to j$)

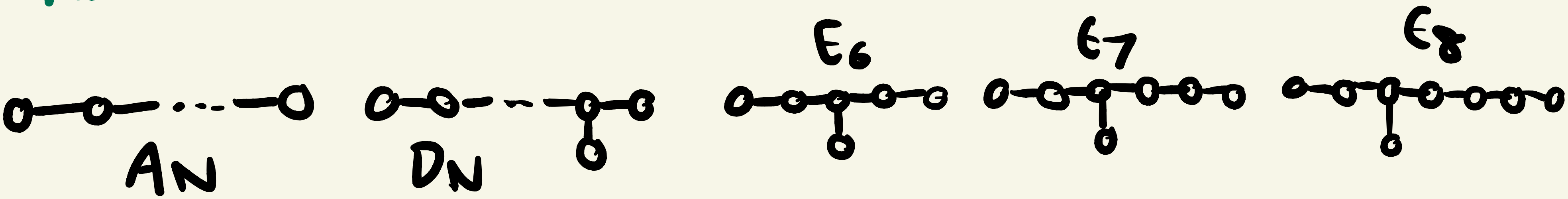
The direct sum of $(X_\bullet, f_\bullet)$ and $(Y_\bullet, g_\bullet)$ is

$$(X_\bullet, f_\bullet) \oplus (Y_\bullet, g_\bullet) \overset{\text{def}}{=} (Z_\bullet, h_\bullet) \text{ where}$$

$$Z_i = X_i \oplus Y_i \text{ and } h_{ij} = f_{ij} \oplus g_{ij}$$

A quiver repn is **indecomposable** if it is **not** isomorphic to the direct sum of two nonzero quiver repns, and is itself nonzero

Thm A quiver has finitely many indecomposable non-isomorphic representations if and only if the undirected graph obtained by the orientation of edges in the quiver is a disjoint union of a finite number of copies of the following Dynkin diagrams:



There is also a complete description of the indecomposable reps (up to $\cong$) in terms of the root system of the associated Dynkin diagram (when the quiver is connected)

Today: Category theory

What is it? "A very flexible and power language", useful for organizing definitions and results, helps us see when specific things are instances of general constructions.

Def. A category $\mathcal{C}$ consists of

1. a "class" of objects $Ob(\mathcal{C})$

   ?

2. a "class" of morphisms $Hom_{\mathcal{C}}(X, Y)$

   for any objects $X, Y \in Ob(\mathcal{C})$.

   Write $f: X \to Y$ to mean $f \in Hom_{\mathcal{C}}(X, Y)$

3. for any objects $X, Y, Z \in Ob(\mathcal{C})$, a composition map $Hom_{\mathcal{C}}(Y, Z) \times Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{C}}(X, Z)$

$$(f, g) \longmapsto f \circ g$$

such that $(f\circ g)\circ h = f\circ(g\circ h)$ for all

$h: X\to Y$ $\quad g: Y\to Z$ and $f: Z\to W$

and such that for each object $X\in Ob(C)$

there is an <u>identity morphism</u> $id_X: X\to X$

such that $\quad f\circ id_X = f$ and $id_X\circ g = g$

whenever these compositions are defined.

The technical distinction between a "class" and a "set" belongs to foundations, nobody pays attention to this.
<u>Upshot</u>: it doesn't matter in practice.

The reason we need to use classes is because otherwise there would be no category of Sets. (As there is no set of all sets)

A set is a class but not every class is a set.

From now on, we write $\boxed{X \in C}$ for $\boxed{X \in Ob(C)}$

Ex. ① Category of Sets with maps as morphisms

② Categories of groups, rings, etc., with homomorphisms as morphisms

③ Category $Vect_k$ of vector spaces $/k$, with linear maps as morphisms

(4) Rep(A) of representations of an algebra A (over a field k), with our usual notion of morphisms as morphisms.

(5) Category of topological spaces, with continuous maps as morphisms.

<u>Def</u> A category C is <u>locally small</u> if the class $Hom_C(X,Y)$ is always a set. All of the examples (1)-(5) have this property.

Let $Aut_C(X)$ consist of all $f: X \to X$ for which there exists $f^{-1}: X \to X$ such that $f \circ f^{-1} = f^{-1} \circ f = id_X : X \to X$.

Fact If C is locally small then $\mathrm{Aut}_C(X)$ is a group for all $X \in C$.

Def A <u>full subcategory</u> of a category C is a category B such that $\mathrm{Ob}(B) \subseteq \mathrm{Ob}(C)$ and $\mathrm{Hom}_B(X,Y) = \mathrm{Hom}_C(X,Y)$ for every $X, Y \in \mathrm{Ob}(B)$.

Ex FVect_k of finite-dim k-vector spaces is a full subcategory of Vect_k.

But Rings is (non-full) subcategory of Abelian Groups

A category D is <u>monoidal</u> if it has a product operation $\otimes : Ob(D) \times Ob(D) \to Ob(D)$ and a unit object 1_D with $1_D \otimes X \cong X \otimes 1_D \cong X$ for all $X \in D$, (where $X \cong Y$ means there are morphisms $f: X \to Y$ and $g: Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$) + some conditions.

<u>Ex</u> Vec_K with $\otimes = \otimes_K$ and $1_{Vec_K} = K$

<u>Ex</u> Abelian Groups with $\otimes = \oplus$ and $1_{AbGroups} = \{0\}$

A category C is <u>enriched</u> over a monoidal category D is for each $X, Y \in C$, the class $\mathrm{Hom}_C(X,Y)$ is an object in D and the composition map $\mathrm{Hom}_C(Y,Z) \times \mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_C(X,Z)$ is an instance of the product $\otimes$ for D.

<u>Ex</u> "locally small" $\equiv$ "enriched over the category of sets"

<u>Ex</u> $\mathrm{Rep}(A)$ for a k-algebra A is enriched over Vect_k

Functors and natural transformations

Let C and D be categories.

Def A *functor* $F : C \to D$ consists of :

1. For each object $X \in C$, an object $F[X] \in D$.
2. For each morphism $\sigma : X \to Y$ in $\mathrm{Hom}_C(X, Y)$, a morphism $F[\sigma] : F[X] \to F[Y]$ in $\mathrm{Hom}_D(F[X], F[Y])$

such that $F[\mathrm{id}_X] = \mathrm{id}_{F[X]}$ and $F[\sigma \circ \sigma'] = F[\sigma] \circ F[\sigma']$

Fact We can compose functors $F : D \to E$ and $G : C \to D$ by setting

$$F \circ G[x] = F[G[x]]$$
$$F \circ G[\sigma] = F[G[\sigma]]$$

Denote composition $F \circ G : C \to E$

Def The identity functor $id_C : C \to C$ has $id_C[x] = x$ and $id_x[\sigma] = \sigma$.

technically a "2-category"

Fact A non-locally small category: the category of categories with functors as morphisms

Ex Suppose C is a locally small category with one object X. Then $\mathrm{Hom}_C(X,X)$ is a monoid. A functor $C \to C$ is the same thing as a monoid homomorphism.

Ex We have "forgetful" functors

Groups $\to$ Sets

Rings $\to$ Abelian Groups

etc.

<u>Ex</u> Define C^{op} to be category with

$$Ob(C) = Ob(C^{op}) \text{ and}$$

$$Hom_C(X,Y) = Hom_{C^{op}}(Y,X).$$

Then vector space duality $*$ can be viewed as a functor $Vect_k \to Vect_k^{op}$

$$V^* = \{\text{linear map } V \to k\}$$

$$(f: V \to W)^* = f^*: W^* \to V^*$$
$$\lambda \mapsto \lambda \circ f$$

Ex If $H \subset G$ are finite groups and $Rep(H)$, $Rep(G)$ are the categories of group repns (over say $\mathbb{C}$), then we can view

$$Ind_H^G : Rep(H) \to Rep(G)$$

a forgetful functor

$$Res_H^G : Rep(G) \to Rep(H)$$

as functors (How do these functors act on morphisms?)

Def Suppose $F : C \to D$ and $G : C \to D$ are functors. A <u>natural transformation</u> $\alpha : F \to G$ consists of, for each $X \in C$, a morphism $\alpha_X : F[X] \to G[X]$ such that the diagram

$$\begin{tikzcd} F[X] \arrow[r, "\alpha_X"] \arrow[d, "{F[\sigma]}"'] & G[X] \arrow[d, "{G[\sigma]}"] \\ F[Y] \arrow[r, "\alpha_Y"] & G[Y] \end{tikzcd}$$

commutes for all morphisms $\sigma : X \to Y$ in $\mathrm{Hom}_C(X, Y)$.

There is an obvious way to compose natural transformations $\alpha : G \to H$ and $\beta : F \to G$ to get $\alpha \circ \beta : F \to H$

A <u>natural isomorphism</u> is a natural transformation $\alpha : F \to G$ for which there exists a natural transformation $\alpha^{-1} : G \to F$ such that

$$(\alpha \circ \alpha^{-1})_X = id_X \quad \text{and} \quad (\alpha^{-1} \circ \alpha)_X = id_X$$

for all $X \in C$

The <u>identity natural transformation</u> $id : F \to F$ that has id_X = identity morphism for all $X \in C$.

Fact The class of all functors $C \to D$ between two categories is itself a category with natural transformations as morphisms.

The notion of when two categories are "the same" is a little subtle. In practice, the following is what is typically used to define this:

Def A functor $F: C \to D$ is an **equivalence of categories** if there exists a functor $G: D \to C$

such that $F \circ G$ and $G \circ F$ are (naturally) isomorphic to the identity functors on C and D.

In this case F and G are called <u>quasi-inverses</u> and C and D are said to be <u>equivalent</u>.